# GRADIENT AS COVECTOR: EXAMPLE IN 2-D

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.2.

One example of a covariant vector is the gradient. As an example, suppose we have a 2-d scalar field given by $\Phi = bxy = br^2\cos\theta\sin\theta$. In rectangular coordinates

$$\frac{\partial\Phi}{\partial x} = by \tag{0.1}$$

$$\frac{\partial\Phi}{\partial y} = bx \tag{0.2}$$

In polar coordinates

$$\frac{\partial\Phi}{\partial r} = 2br\cos\theta\sin\theta \tag{0.3}$$

$$\frac{\partial\Phi}{\partial\theta} = br^2\left(\cos^2\theta - \sin^2\theta\right) \tag{0.4}$$

Note that because we have absorbed the factor of $r$ needed for an incremental displacement in the $\theta$ direction into the basis vector $\mathbf{e}_\theta$, there is no extra factor of $1/r$ in the $\frac{\partial\Phi}{\partial\theta}$ term, as there would be if we had used unit basis vectors.

Now suppose we have a vector $v^i$ with components given in rectangular coordinates. Then the scalar product is

$$v^i\partial_i\Phi = byv^x + bxv^y \tag{0.5}$$

If we convert $v$ to polar coords, then

$$v^r = v^x\cos\theta + v^y\sin\theta \tag{0.6}$$

$$v^\theta = -v^x\frac{\sin\theta}{r} + v^y\frac{\cos\theta}{r} \tag{0.7}$$

The scalar product now is

(0.8)
$$v^i\partial_i\Phi = \left(v^x\cos\theta + v^y\sin\theta\right)\left(2br\cos\theta\sin\theta\right) + \left(-v^x\frac{\sin\theta}{r} + v^y\frac{\cos\theta}{r}\right)br^2\left(\cos^2\theta - \sin^2\theta\right)$$
(0.9)
$$= brv^x\sin\theta\left(2\cos^2\theta - \cos^2\theta + \sin^2\theta\right) + brv^y\cos\theta\left(2\sin^2\theta + \cos^2\theta - \sin^2\theta\right)$$
(0.10)
$$= brv^x\sin\theta + brv^y\cos\theta$$
(0.11)
$$= byv^x + bxv^y$$

Thus the scalar product is invariant.