

GRADIENT AS COVECTOR: EXAMPLE IN 2-D

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.2.

One example of a covariant vector is the gradient. As an example, suppose we have a 2-d scalar field given by $\Phi = bxy = br^2 \cos \theta \sin \theta$. In rectangular coordinates

$$\frac{\partial \Phi}{\partial x} = by \quad (1)$$

$$\frac{\partial \Phi}{\partial y} = bx \quad (2)$$

In polar coordinates

$$\frac{\partial \Phi}{\partial r} = 2br \cos \theta \sin \theta \quad (3)$$

$$\frac{\partial \Phi}{\partial \theta} = br^2 (\cos^2 \theta - \sin^2 \theta) \quad (4)$$

Note that because we have absorbed the factor of r needed for an incremental displacement in the θ direction into the basis vector \mathbf{e}_θ , there is no extra factor of $1/r$ in the $\frac{\partial \Phi}{\partial \theta}$ term, as there would be if we had used unit basis vectors.

Now suppose we have a vector v^i with components given in rectangular coordinates. Then the scalar product is

$$v^i \partial_i \Phi = byv^x + bxv^y \quad (5)$$

If we convert v to polar coords, then

$$v^r = v^x \cos \theta + v^y \sin \theta \quad (6)$$

$$v^\theta = -v^x \frac{\sin \theta}{r} + v^y \frac{\cos \theta}{r} \quad (7)$$

The scalar product now is

$$v^i \partial_i \Phi = (v^x \cos \theta + v^y \sin \theta) (2br \cos \theta \sin \theta) + \left(-v^x \frac{\sin \theta}{r} + v^y \frac{\cos \theta}{r} \right) br^2 (\cos^2 \theta - \sin^2 \theta) \quad (8)$$

$$= brv^x \sin \theta (2 \cos^2 \theta - \cos^2 \theta + \sin^2 \theta) + brv^y \cos \theta (2 \sin^2 \theta + \cos^2 \theta - \sin^2 \theta) \quad (9)$$

$$= brv^x \sin \theta + brv^y \cos \theta \quad (10)$$

$$= byv^x + bxv^y \quad (11)$$

Thus the scalar product is invariant.