

TENSOR PRODUCT: NUMERICAL EXAMPLE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.4.

It's useful to do a tensor calculation using actual numbers, just to see how they work. Suppose we have a vector with components

$$A^i = [1, 2, -1, 0] \quad (1)$$

where the units are metres, and a second vector given by

$$B^i = [3, -1, 0, -2] \quad (2)$$

where the units are seconds⁻¹.

Since the units are different (even using relativistic units where length = time, since vector B^i has units of 1/time), we cannot add these vectors. However, we can multiply them to get a rank-2 tensor:

$$M^{ij} = A^i B^j = \begin{bmatrix} 3 & -1 & 0 & -2 \\ 6 & -2 & 0 & -4 \\ -3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3)$$

The first row consists of A^0 multiplied by each element of B in turn, and so on (the indices i and j each run for 0 to 3). The units of each element of M^{ij} are $\text{m} \cdot \text{s}^{-1}$.

The question in Moore asks if this product is commutative, which isn't really the right thing to ask, since obviously $A^i B^j = B^j A^i$. However, the product is not *symmetric*, in the sense that $A^i B^j \neq A^j B^i$. That is, it matters which vector is used to determine the row index and which the column index.

The trace of M is given by

$$M^i_i = g_{ij} M^{ij} \quad (4)$$

$$= g_{ij} A^i B^j \quad (5)$$

This is just the scalar product $\mathbf{A} \cdot \mathbf{B}$. If we use the special relativity metric η_{ij} , then

$$\eta_{ij}A^iB^j = -(1 \times 3) + (-1 \times 2) + (-1 \times 0) + (-2 \times 0) = -5 \quad (6)$$

We could also have obtained this result from the matrix M^{ij} directly by calculating $\eta_{00}M^{00} + \eta_{11}M^{11} = -5$ (the other two diagonal elements are zero).