TENSOR PRODUCT: NUMERICAL EXAMPLE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.4.

It's useful to do a tensor calculation using actual numbers, just to see how they work. Suppose we have a vector with components

$$A^{i}=[1,2,-1,0] \tag{1}$$

where the units are metres, and a second vector given by

$$B^{i}=[3,-1,0,-2] \tag{2}$$

where the units are seconds^{-1}.

Since the units are different (even using relativistic units where length = time, since vector B^{i} has units of 1/time), we cannot add these vectors. However, we can multiply them to get a rank-2 tensor:

$$M^{ij}=A^{i}B^{j}=\left[\begin{array}{cccc}3 & -1 & 0 & -2\\ 6 & -2 & 0 & -4\\ -3 & 1 & 0 & 2\\ 0 & 0 & 0 & 0\end{array}\right] \tag{3}$$

The first row consists of A^{0} multiplied by each element of B in turn, and so on (the indices i and j each run for 0 to 3). The units of each element of M^{ij} are $\mathrm{m}\cdot\mathrm{s}^{-1}$.

The question in Moore asks if this product is commutative, which isn't really the right thing to ask, since obviously $A^{i}B^{j}=B^{j}A^{i}$. However, the product is not *symmetric*, in the sense that $A^{i}B^{j}\neq A^{j}B^{i}$. That is, it matters which vector is used to determine the row index and which the column index.

The trace of M is given by

$$\begin{aligned} M^{i}{}_{i} &= g_{ij}M^{ij} && (4)\\ &= g_{ij}A^{i}B^{j} && (5)\end{aligned}$$

This is just the scalar product $\mathbf{A}\cdot\mathbf{B}$. If we use the special relativity metric η_{ij}, then

$$\eta_{ij}A^{i}B^{j}=-(1\times3)+(-1\times2)+(-1\times0)+(-2\times0)=-5 \tag{6}$$

We could also have obtained this result from the matrix M^{ij} directly by calculating $\eta_{00}M^{00}+\eta_{11}M^{11}=-5$ (the other two diagonal elements are zero).