TENSORS: SYMMETRIC AND ANTI-SYMMETRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.6.

Tensors, like matrices, can be symmetric or anti-symmetric. Since a tensor can have a rank higher than 2, however, a single tensor can have more than one symmetry. For a rank-2 tensor T^{ij}, it is symmetric if $T^{ij}=T^{ji}$ and anti-symmetric if $T^{ij}=-T^{ji}$. In matrix terminology, a symmetric rank-2 tensor is equal to its transpose, and an anti-symmetric rank-2 tensor is equal to the negative of its transpose.

A higher rank tensor can be symmetric or anti-symmetric in any pair of its indices, provided both indices are either upper or lower. For example, $F^{ijk}{}_{lm}$ can be symmetric or anti-symmetric in any pair selected from i,j,k or in the pair l,m, but not in one upper and one lower index. Thus if $F^{ijk}{}_{lm}=F^{kji}{}_{lm}$ and $F^{ijk}{}_{lm}=-F^{ijk}{}_{ml}$, then $F^{ijk}{}_{lm}$ is symmetric in i and k, and anti-symmetric in l and m.

Returning to rank-2 tensors, we can show that the symmetry property is an invariant:

$$F'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}F^{kl} \tag{1}$$

$$= \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}F^{lk} \tag{2}$$

$$= F'^{ji} \tag{3}$$

If a tensor is symmetric in a pair of upper indices, then if both indices are lowered, the resulting tensor is also symmetric in the two lower indices:

$$F_{ij} = g_{ik}g_{jl}F^{kl} \tag{4}$$

$$= g_{ik}g_{jl}F^{lk} \tag{5}$$

$$= F_{ji} \tag{6}$$

Similarly, the anti-symmetric property persists through lowering of indices. If $T^{ij}=-T^{ji}$

$$T_{ij} = g_{ik}g_{jl}T^{kl} \tag{7}$$
$$= -g_{ik}g_{jl}T^{lk} \tag{8}$$
$$= -T_{ji} \tag{9}$$

If $T^{ij} = -T^{ji}$ then all diagonal elements must be zero, since $T^{ii} = -T^{ii}$ has only zero as a solution. Also, the trace is

$$T^{i}_{\ i} = g_{ij}T^{ij} \tag{10}$$
$$= -g_{ij}T^{ji} \tag{11}$$
$$= -g_{ji}T^{ji} \tag{12}$$
$$= -T^{i}_{\ i} \tag{13}$$

In line 3, we used $g_{ij} = g_{ji}$, since in terms of the basis vectors, $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$, and thus the metric tensor is symmetric. Thus the trace is also zero for an anti-symmetric tensor.

A rank 2 symmetric tensor in n dimensions has all the diagonal elements and the upper (or lower) triangular set of elements as independent components, so the total number of independent elements is $1+2+\ldots+n = \frac{1}{2}n(n+1)$. An anti-symmetric tensor has zeroes on the diagonal, so it has $\frac{1}{2}n(n+1) - n = \frac{1}{2}n(n-1)$ independent elements.