

TENSORS: SYMMETRIC AND ANTI-SYMMETRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.6.

Tensors, like matrices, can be symmetric or anti-symmetric. Since a tensor can have a rank higher than 2, however, a single tensor can have more than one symmetry. For a rank-2 tensor T^{ij} , it is symmetric if $T^{ij} = T^{ji}$ and anti-symmetric if $T^{ij} = -T^{ji}$. In matrix terminology, a symmetric rank-2 tensor is equal to its transpose, and an anti-symmetric rank-2 tensor is equal to the negative of its transpose.

A higher rank tensor can be symmetric or anti-symmetric in any pair of its indices, provided both indices are either upper or lower. For example, F^{ijk}_{lm} can be symmetric or anti-symmetric in any pair selected from i, j, k or in the pair l, m , but not in one upper and one lower index. Thus if $F^{ijk}_{lm} = F^{kji}_{lm}$ and $F^{ijk}_{lm} = -F^{ijk}_{ml}$, then F^{ijk}_{lm} is symmetric in i and k , and anti-symmetric in l and m .

Returning to rank-2 tensors, we can show that the symmetry property is an invariant:

$$F'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} F^{kl} \quad (1)$$

$$= \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} F^{lk} \quad (2)$$

$$= F'^{ji} \quad (3)$$

If a tensor is symmetric in a pair of upper indices, then if both indices are lowered, the resulting tensor is also symmetric in the two lower indices:

$$F_{ij} = g_{ik} g_{jl} F^{kl} \quad (4)$$

$$= g_{ik} g_{jl} F^{lk} \quad (5)$$

$$= F_{ji} \quad (6)$$

Similarly, the anti-symmetric property persists through lowering of indices. If $T^{ij} = -T^{ji}$

$$T_{ij} = g_{ik}g_{jl}T^{kl} \quad (7)$$

$$= -g_{ik}g_{jl}T^{lk} \quad (8)$$

$$= -T_{ji} \quad (9)$$

If $T^{ij} = -T^{ji}$ then all diagonal elements must be zero, since $T^{ii} = -T^{ii}$ has only zero as a solution. Also, the trace is

$$T^i_i = g_{ij}T^{ij} \quad (10)$$

$$= -g_{ij}T^{ji} \quad (11)$$

$$= -g_{ji}T^{ji} \quad (12)$$

$$= -T^i_i \quad (13)$$

In line 3, we used $g_{ij} = g_{ji}$, since in terms of the basis vectors, $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$, and thus the metric tensor is symmetric. Thus the trace is also zero for an anti-symmetric tensor.

A rank 2 symmetric tensor in n dimensions has all the diagonal elements and the upper (or lower) triangular set of elements as independent components, so the total number of independent elements is $1 + 2 + \dots + n = \frac{1}{2}n(n+1)$. An anti-symmetric tensor has zeroes on the diagonal, so it has $\frac{1}{2}n(n+1) - n = \frac{1}{2}n(n-1)$ independent elements.