TRANSFORMING DERIVATIVES OF FOUR-VECTORS AND SCALARS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.7.

In special relativity, a general four-vector $\mathbf{A}$ transforms like

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j \tag{1}$$

If we take the derivative of $\mathbf{A}$with respect to proper time, and assume that $\mathbf{A}$ is a function of position, then

$$\frac{dA^i}{d\tau} = \frac{\partial A^i}{\partial x^j}\frac{\partial x^j}{\partial \tau} \tag{2}$$

How does this transform? We get

$$\begin{aligned}
\frac{dA'^i}{d\tau} &= \frac{\partial A'^i}{\partial x'^k}\frac{\partial x'^k}{\partial \tau} && (3)\\
&= \frac{\partial}{\partial x'^k}\left(\frac{\partial x'^i}{\partial x^j}A^j\right)\frac{\partial x'^k}{\partial \tau} && (4)\\
&= \left[\frac{\partial^2 x'^i}{\partial x'^k \partial x^j}A^j + \frac{\partial x'^i}{\partial x^j}\frac{\partial A^j}{\partial x'^k}\right]\frac{\partial x'^k}{\partial \tau} && (5)\\
&= \frac{\partial^2 x'^i}{\partial x'^k \partial x^j}A^j\frac{\partial x'^k}{\partial \tau} + \frac{\partial x'^i}{\partial x^j}\frac{dA^j}{d\tau} && (6)
\end{aligned}$$

In order for $\frac{dA^i}{d\tau}$ to be a four-vector, the first term in the last line would need to be zero. Since A^j and $x'^k(\tau)$ are arbitrary this means that

$$\frac{\partial}{\partial x'^k}\left(\frac{\partial x'^i}{\partial x^j}\right) = 0 \tag{7}$$

That is, the coordinate partial derivatives must be independent of position. In general this isn't true, but in flat space, coordinate transformations are given by the Lorentz transformations which are independent of position, so in flat space, $\frac{dA^i}{d\tau}$ is a four-vector.

If we have a scalar function of position $\Phi\left(x^i\right)$, then

$$\frac{d\Phi}{d\tau}=\frac{\partial\Phi}{\partial x^{j}}\frac{\partial x^{j}}{\partial\tau} \tag{8}$$

Doing the transform, we get

$$\frac{d\Phi'}{d\tau}=\frac{\partial\Phi'}{\partial x'^{k}}\frac{\partial x'^{k}}{\partial\tau} \tag{9}$$

However, since scalars are invariant $\Phi'=\Phi$ so

$$\frac{d\Phi'}{d\tau} = \frac{\partial\Phi}{\partial x'^{k}}\frac{\partial x'^{k}}{\partial\tau} \tag{10}$$

$$= \frac{d\Phi}{d\tau} \tag{11}$$

So in this case, the derivative of a scalar field is a valid scalar under transformation.

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