INERTIA TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.11.

In basic physics, the moment of inertia is usually defined by the integral

$$I=\int_V r^2\rho(\mathbf{r})\,d^3\mathbf{r} \tag{1}$$

where r is the perpendicular distance of the volume element from the axis of rotation and ρ is the mass density. In fact, this is true only for special cases, where the mass distribution is symmetric with respect to the axis (cases such as a sphere, cylinder, etc). In rotational motion, the angular momentum of a spinning rigid body (that is, a body which does not deform as it moves) is given by $\mathbf{L}=I\omega$ where ω is the angular velocity vector.

In the more general case, where the object isn't symmetric, the moment of inertia becomes a tensor, and the angular momentum equation becomes (in 3-d):

$$L^i=I^{ij}\omega_j \tag{2}$$

The derivation of this tensor would take us too far afield here, but basically, the off-diagonal elements of I^{ij} measure the amount of asymmetry in various directions. For example, in rectangular coordinates

$$I^{xy}=I^{yx}=\int xy\rho(\mathbf{r})\,d^3\mathbf{r} \tag{3}$$

where the coordinates are measured relative to the centre of mass. For a symmetric object, this integral is always zero, since any mass element at $\mathbf{r}$ is balanced by an equal mass element at $-\mathbf{r}$.

If we start with a standard 3-d rectangular system and rotate it by an angle θ about the z axis to get the primed system, then the transformation can be found by considering the projection of the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ onto the $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$ axes.

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}'\cos\theta-\hat{\mathbf{y}}'\sin\theta \tag{4}$$
$$\hat{\mathbf{y}} = \hat{\mathbf{x}}'\sin\theta+\hat{\mathbf{y}}'\cos\theta \tag{5}$$

By taking the scalar product of both these equations with $\hat{\mathbf{x}}'$ we can get the components of $\hat{\mathbf{x}}'$ along the original axes, and similarly for $\hat{\mathbf{y}}'$. For example $\hat{\mathbf{x}}' \cdot \hat{\mathbf{x}} = \hat{\mathbf{x}}' \cdot (\hat{\mathbf{x}}' \cos\theta - \hat{\mathbf{y}}' \sin\theta) = \cos\theta$ is the component of $\hat{\mathbf{x}}'$ in the $\hat{\mathbf{x}}$ direction. We get

$$\begin{aligned} \hat{\mathbf{x}}' &= \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta && (6) \\ \hat{\mathbf{y}}' &= -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta && (7) \end{aligned}$$

If we now take an arbitrary vector $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ and take its scalar product with $\hat{\mathbf{x}}'$, we get

$$\begin{aligned} \mathbf{r}\cdot\hat{\mathbf{x}}' &= x\cos\theta + y\sin\theta && (8) \\ &= x' && (9) \end{aligned}$$

Similarly for y':

$$y' = -x\sin\theta + y\cos\theta \tag{10}$$

The transformation partials are then

$$R^i{}_j = \frac{\partial x'^i}{\partial x^j} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

Now suppose we have an inertia tensor that is diagonal (this is true for symmetric objects, but it is always possible to find some *principal axes* where this is true for any object) so that

$$I^{ij} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{12}$$

We can transform this to the rotated system using the usual tensor transformation rule:

$$I'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l} I^{kl} = \begin{bmatrix} I_1\cos^2\theta + I_2\sin^2\theta & (I_2 - I_1)\cos\theta\sin\theta & 0 \\ (I_2 - I_1)\cos\theta\sin\theta & I_2\cos^2\theta + I_1\sin^2\theta & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{13}$$

This is equivalent to a matrix product as we can see by taking it in stages. First consider the intermediate product P^{il}

$$P^{il} = \frac{\partial x'^i}{\partial x^k} I^{kl} = R^i{}_k I^{kl} \tag{14}$$

This is the sum over the product of elements in the ith row of R with elements in the lth column of I. Thus this is a matrix product with the factors in the order RI. Now consider

$$I'^{ij} = P^{il}\frac{\partial x'^{j}}{\partial x^{l}} \tag{15}$$

This time, we're summing over the product of elements in the ith row of P with elements in the jth *row* of $\frac{\partial x'^{j}}{\partial x^{l}}$. In order to make this a matrix product, we have to sum over the row elements of one matrix multiplied by the column elements of the second matrix, so we need to take the transpose of $\frac{\partial x'^{j}}{\partial x^{l}}$ to make this work. That is

$$I' = PR^{T} = RIR^{T} \tag{16}$$