

## INERTIA TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.11.

In basic physics, the moment of inertia is usually defined by the integral

$$I = \int_V r^2 \rho(\mathbf{r}) d^3\mathbf{r} \quad (1)$$

where  $r$  is the perpendicular distance of the volume element from the axis of rotation and  $\rho$  is the mass density. In fact, this is true only for special cases, where the mass distribution is symmetric with respect to the axis (cases such as a sphere, cylinder, etc). In rotational motion, the angular momentum of a spinning rigid body (that is, a body which does not deform as it moves) is given by  $\mathbf{L} = I\boldsymbol{\omega}$  where  $\boldsymbol{\omega}$  is the angular velocity vector.

In the more general case, where the object isn't symmetric, the moment of inertia becomes a tensor, and the angular momentum equation becomes (in 3-d):

$$L^i = I^{ij} \omega_j \quad (2)$$

The derivation of this tensor would take us too far afield here, but basically, the off-diagonal elements of  $I^{ij}$  measure the amount of asymmetry in various directions. For example, in rectangular coordinates

$$I^{xy} = I^{yx} = \int xy\rho(\mathbf{r}) d^3\mathbf{r} \quad (3)$$

where the coordinates are measured relative to the centre of mass. For a symmetric object, this integral is always zero, since any mass element at  $\mathbf{r}$  is balanced by an equal mass element at  $-\mathbf{r}$ .

If we start with a standard 3-d rectangular system and rotate it by an angle  $\theta$  about the  $z$  axis to get the primed system, then the transformation can be found by considering the projection of the unit vectors  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  onto the  $\hat{\mathbf{x}}'$  and  $\hat{\mathbf{y}}'$  axes.

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}' \cos \theta - \hat{\mathbf{y}}' \sin \theta \quad (4)$$

$$\hat{\mathbf{y}} = \hat{\mathbf{x}}' \sin \theta + \hat{\mathbf{y}}' \cos \theta \quad (5)$$

By taking the scalar product of both these equations with  $\hat{\mathbf{x}}'$  we can get the components of  $\hat{\mathbf{x}}'$  along the original axes, and similarly for  $\hat{\mathbf{y}}'$ . For example  $\hat{\mathbf{x}}' \cdot \hat{\mathbf{x}} = \hat{\mathbf{x}}' \cdot (\hat{\mathbf{x}}' \cos \theta - \hat{\mathbf{y}}' \sin \theta) = \cos \theta$  is the component of  $\hat{\mathbf{x}}'$  in the  $\hat{\mathbf{x}}$  direction. We get

$$\hat{\mathbf{x}}' = \hat{\mathbf{x}} \cos \theta + \hat{\mathbf{y}} \sin \theta \quad (6)$$

$$\hat{\mathbf{y}}' = -\hat{\mathbf{x}} \sin \theta + \hat{\mathbf{y}} \cos \theta \quad (7)$$

If we now take an arbitrary vector  $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$  and take its scalar product with  $\hat{\mathbf{x}}'$ , we get

$$\mathbf{r} \cdot \hat{\mathbf{x}}' = x \cos \theta + y \sin \theta \quad (8)$$

$$= x' \quad (9)$$

Similarly for  $y'$ :

$$y' = -x \sin \theta + y \cos \theta \quad (10)$$

The transformation partials are then

$$R^i_j = \frac{\partial x'^i}{\partial x^j} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (11)$$

Now suppose we have an inertia tensor that is diagonal (this is true for symmetric objects, but it is always possible to find some *principal axes* where this is true for any object) so that

$$I^{ij} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \quad (12)$$

We can transform this to the rotated system using the usual tensor transformation rule:

$$I'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} I^{kl} = \begin{bmatrix} I_1 \cos^2 \theta + I_2 \sin^2 \theta & (I_2 - I_1) \cos \theta \sin \theta & 0 \\ (I_2 - I_1) \cos \theta \sin \theta & I_2 \cos^2 \theta + I_1 \sin^2 \theta & 0 \\ 0 & 0 & I_3 \end{bmatrix} \quad (13)$$

This is equivalent to a matrix product as we can see by taking it in stages. First consider the intermediate product  $P^{il}$

$$P^{il} = \frac{\partial x'^i}{\partial x^k} I^{kl} = R^i_k I^{kl} \quad (14)$$

This is the sum over the product of elements in the  $i$ th row of  $R$  with elements in the  $l$ th column of  $I$ . Thus this is a matrix product with the factors in the order  $RI$ . Now consider

$$I'^{ij} = P^{il} \frac{\partial x'^j}{\partial x^l} \quad (15)$$

This time, we're summing over the product of elements in the  $i$ th row of  $P$  with elements in the  $j$ th row of  $\frac{\partial x'^j}{\partial x^l}$ . In order to make this a matrix product, we have to sum over the row elements of one matrix multiplied by the column elements of the second matrix, so we need to take the transpose of  $\frac{\partial x'^j}{\partial x^l}$  to make this work. That is

$$I' = PR^T = RIR^T \quad (16)$$