

INERTIA TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.11.

In basic physics, the moment of inertia is usually defined by the integral

$$(0.1) \quad I = \int_V r^2 \rho(\mathbf{r}) d^3\mathbf{r}$$

where r is the perpendicular distance of the volume element from the axis of rotation and ρ is the mass density. In fact, this is true only for special cases, where the mass distribution is symmetric with respect to the axis (cases such as a sphere, cylinder, etc). In rotational motion, the angular momentum of a spinning rigid body (that is, a body which does not deform as it moves) is given by $\mathbf{L} = I\boldsymbol{\omega}$ where $\boldsymbol{\omega}$ is the angular velocity vector.

In the more general case, where the object isn't symmetric, the moment of inertia becomes a tensor, and the angular momentum equation becomes (in 3-d):

$$(0.2) \quad L^i = I^{ij} \omega_j$$

The derivation of this tensor would take us too far afield here, but basically, the off-diagonal elements of I^{ij} measure the amount of asymmetry in various directions. For example, in rectangular coordinates

$$(0.3) \quad I^{xy} = I^{yx} = \int xy \rho(\mathbf{r}) d^3\mathbf{r}$$

where the coordinates are measured relative to the centre of mass. For a symmetric object, this integral is always zero, since any mass element at \mathbf{r} is balanced by an equal mass element at $-\mathbf{r}$.

If we start with a standard 3-d rectangular system and rotate it by an angle θ about the z axis to get the primed system, then the transformation can be found by considering the projection of the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ onto the $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$ axes.

$$(0.4) \quad \hat{\mathbf{x}} = \hat{\mathbf{x}}' \cos \theta - \hat{\mathbf{y}}' \sin \theta$$

$$(0.5) \quad \hat{\mathbf{y}} = \hat{\mathbf{x}}' \sin \theta + \hat{\mathbf{y}}' \cos \theta$$

By taking the scalar product of both these equations with $\hat{\mathbf{x}}'$ we can get the components of $\hat{\mathbf{x}}'$ along the original axes, and similarly for $\hat{\mathbf{y}}'$. For example $\hat{\mathbf{x}}' \cdot \hat{\mathbf{x}} = \hat{\mathbf{x}}' \cdot (\hat{\mathbf{x}}' \cos \theta - \hat{\mathbf{y}}' \sin \theta) = \cos \theta$ is the component of $\hat{\mathbf{x}}'$ in the $\hat{\mathbf{x}}$ direction. We get

$$(0.6) \quad \hat{\mathbf{x}}' = \hat{\mathbf{x}} \cos \theta + \hat{\mathbf{y}} \sin \theta$$

$$(0.7) \quad \hat{\mathbf{y}}' = -\hat{\mathbf{x}} \sin \theta + \hat{\mathbf{y}} \cos \theta$$

If we now take an arbitrary vector $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ and take its scalar product with $\hat{\mathbf{x}}'$, we get

$$(0.8) \quad \mathbf{r} \cdot \hat{\mathbf{x}}' = x \cos \theta + y \sin \theta$$

$$(0.9) \quad = x'$$

Similarly for y' :

$$(0.10) \quad y' = -x \sin \theta + y \cos \theta$$

The transformation partials are then

$$(0.11) \quad R^i_j = \frac{\partial x'^i}{\partial x^j} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now suppose we have an inertia tensor that is diagonal (this is true for symmetric objects, but it is always possible to find some *principal axes* where this is true for any object) so that

$$(0.12) \quad I^{ij} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

We can transform this to the rotated system using the usual tensor transformation rule:

$$(0.13) \quad I'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} I^{kl} = \begin{bmatrix} I_1 \cos^2 \theta + I_2 \sin^2 \theta & (I_2 - I_1) \cos \theta \sin \theta & 0 \\ (I_2 - I_1) \cos \theta \sin \theta & I_2 \cos^2 \theta + I_1 \sin^2 \theta & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

This is equivalent to a matrix product as we can see by taking it in stages. First consider the intermediate product P^{il}

$$(0.14) \quad P^{il} = \frac{\partial x'^i}{\partial x^k} I^{kl} = R^i_k I^{kl}$$

This is the sum over the product of elements in the i th row of R with elements in the l th column of I . Thus this is a matrix product with the factors in the order RI . Now consider

$$(0.15) \quad I'^{ij} = P^{il} \frac{\partial x'^j}{\partial x^l}$$

This time, we're summing over the product of elements in the i th row of P with elements in the j th row of $\frac{\partial x'^j}{\partial x^l}$. In order to make this a matrix product, we have to sum over the row elements of one matrix multiplied by the column elements of the second matrix, so we need to take the transpose of $\frac{\partial x'^j}{\partial x^l}$ to make this work. That is

$$(0.16) \quad I' = PR^T = RIR^T$$