ELECTROMAGNETIC FIELD TENSOR: LORENTZ TRANSFORMATIONS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 7; Problems 7.1.

Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.48.

To apply this to Griffiths problem 12.48, use

$$t^{ij}=\left[\begin{array}{cccc}0 & t^{01} & t^{02} & t^{03}\\ -t^{01} & 0 & t^{12} & t^{13}\\ -t^{02} & -t^{12} & 0 & t^{23}\\ -t^{03} & -t^{13} & -t^{23} & 0\end{array}\right] \tag{1}$$

in place of F^{ij} in what follows. The results are the same.

The electromagnetic field tensor is

$$F^{ij}=\left[\begin{array}{cccc}0 & E_x & E_y & E_z\\ -E_x & 0 & B_z & -B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0\end{array}\right] \tag{2}$$

We can use the usual tensor transformation rules to see how the electric and magnetic fields transform under a Lorentz transformation. We get

$$F'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}F^{kl} \tag{3}$$

$$= \Lambda^i{}_k\Lambda^j{}_lF^{kl} \tag{4}$$

where the Lorentz transformation matrix is

$$\Lambda^i{}_k=\left[\begin{array}{cccc}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{5}$$

As we saw when discussing the inertia tensor, we can write this transformation as a matrix equation

$$F' = \Lambda F \Lambda^T \tag{6}$$

The first product is

$$\Lambda F = \begin{bmatrix} \gamma\beta E_x & \gamma E_x & \gamma E_y - \gamma\beta B_z & \gamma E_z + \gamma\beta B_y \\ -\gamma E_x & -\gamma\beta E_x & -\gamma\beta E_y + \gamma B_z & -\gamma\beta E_z - \gamma B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \tag{7}$$

The final product is

$$F' = \Lambda F \Lambda^T = \begin{bmatrix} 0 & \gamma^2\left(1-\beta^2\right)E_x & \gamma E_y - \gamma\beta B_z & \gamma E_z + \gamma\beta B_y \\ -\gamma^2\left(1-\beta^2\right)E_x & 0 & -\gamma\beta E_y + \gamma B_z & -\gamma\beta E_z - \gamma B_y \\ -\gamma E_y + \gamma\beta B_z & \gamma\beta E_y - \gamma B_z & 0 & B_x \\ -\gamma E_z - \gamma\beta B_y & \gamma\beta E_z + \gamma B_y & -B_x & 0 \end{bmatrix} \tag{8}$$

Using $\gamma = 1/\sqrt{1-\beta^2}$ we get

$$F' = \begin{bmatrix} 0 & E_x & \gamma E_y - \gamma\beta B_z & \gamma E_z + \gamma\beta B_y \\ -E_x & 0 & -\gamma\beta E_y + \gamma B_z & -\gamma\beta E_z - \gamma B_y \\ -\gamma E_y + \gamma\beta B_z & \gamma\beta E_y - \gamma B_z & 0 & B_x \\ -\gamma E_z - \gamma\beta B_y & \gamma\beta E_z + \gamma B_y & -B_x & 0 \end{bmatrix} \tag{9}$$

From this, we see that

$$E'_x = E_x \tag{10}$$
$$E'_y = \gamma E_y - \gamma\beta B_z \tag{11}$$
$$E'_z = \gamma E_z + \gamma\beta B_y \tag{12}$$
$$B'_x = B_x \tag{13}$$
$$B'_y = \gamma\beta E_z + \gamma B_y \tag{14}$$
$$B'_z = -\gamma\beta E_y + \gamma B_z \tag{15}$$

Unlike lengths, the components of E and B in the direction of motion are unchanged, while those perpendicular to the motion are altered.

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