

## ELECTROMAGNETIC FIELD TENSOR: LORENTZ TRANSFORMATIONS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 7; Problems 7.1.

Griffiths, David J. (2007), *Introduction to Electrodynamics*, 3rd Edition; Pearson Education - Chapter 12, Problem 12.48.

To apply this to Griffiths problem 12.48, use

$$(1) \quad t^{ij} = \begin{bmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{bmatrix}$$

in place of  $F^{ij}$  in what follows. The results are the same.  
The electromagnetic field tensor is

$$(2) \quad F^{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

We can use the usual tensor transformation rules to see how the electric and magnetic fields transform under a Lorentz transformation. We get

$$(3) \quad F'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} F^{kl}$$

$$(4) \quad = \Lambda^i_k \Lambda^j_l F^{kl}$$

where the Lorentz transformation matrix is

$$(5) \quad \Lambda^i_k = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As we saw when discussing the inertia tensor, we can write this transformation as a matrix equation

$$(6) \quad F' = \Lambda F \Lambda^T$$

The first product is

$$(7) \quad \Lambda F = \begin{bmatrix} \gamma\beta E_x & \gamma E_x & \gamma E_y - \gamma\beta B_z & \gamma E_z + \gamma\beta B_y \\ -\gamma E_x & -\gamma\beta E_x & -\gamma\beta E_y + \gamma B_z & -\gamma\beta E_z - \gamma B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

The final product is

$$(8) \quad F' = \Lambda F \Lambda^T = \begin{bmatrix} 0 & \gamma^2(1-\beta^2)E_x & \gamma E_y - \gamma\beta B_z & \gamma E_z + \gamma\beta B_y \\ -\gamma^2(1-\beta^2)E_x & 0 & -\gamma\beta E_y + \gamma B_z & -\gamma\beta E_z - \gamma B_y \\ -\gamma E_y + \gamma\beta B_z & \gamma\beta E_y - \gamma B_z & 0 & B_x \\ -\gamma E_z - \gamma\beta B_y & \gamma\beta E_z + \gamma B_y & -B_x & 0 \end{bmatrix}$$

Using  $\gamma = 1/\sqrt{1-\beta^2}$  we get

$$(9) \quad F' = \begin{bmatrix} 0 & E_x & \gamma E_y - \gamma\beta B_z & \gamma E_z + \gamma\beta B_y \\ -E_x & 0 & -\gamma\beta E_y + \gamma B_z & -\gamma\beta E_z - \gamma B_y \\ -\gamma E_y + \gamma\beta B_z & \gamma\beta E_y - \gamma B_z & 0 & B_x \\ -\gamma E_z - \gamma\beta B_y & \gamma\beta E_z + \gamma B_y & -B_x & 0 \end{bmatrix}$$

From this, we see that

$$(10) \quad E'_x = E_x$$

$$(11) \quad E'_y = \gamma E_y - \gamma\beta B_z$$

$$(12) \quad E'_z = \gamma E_z + \gamma\beta B_y$$

$$(13) \quad B'_x = B_x$$

$$(14) \quad B'_y = \gamma\beta E_z + \gamma B_y$$

$$(15) \quad B'_z = -\gamma\beta E_y + \gamma B_z$$

Unlike lengths, the components of  $E$  and  $B$  in the direction of motion are unchanged, while those perpendicular to the motion are altered.

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