ELECTROMAGNETIC FIELD TENSOR: FOUR-POTENTIAL

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 7; Problem 7.3.

We've seen that we can arrive at the electromagnetic field tensor F^{ij} by a bit of educated guesswork, and that the result is

$$F^{ij}=\left[\begin{array}{cccc} 0 & E_{x} & E_{y} & E_{z}\\ -E_{x} & 0 & B_{z} & -B_{y}\\ -E_{y} & -B_{z} & 0 & B_{x}\\ -E_{z} & B_{y} & -B_{x} & 0 \end{array}\right] \tag{1}$$

What happens if we try to generalize the relation between electric field and electric potential to tensor form? That is, we start with the electrostatic relation

$$\mathbf{E}=-\nabla\phi \tag{2}$$

and generalize this to tensor form using F^{ij}. Since F^{ij} contains the components of the electric field it's natural to try replacing the LHS with F^{ij}. The RHS suggests that we introduce a four-vector A^{j} and take its derivative, so our first attempt looks like $F^{ij}=\partial^{i}A^{j}$. However, F^{ij} must be anti-symmetric, which $\partial^{i}A^{j}$ clearly isn't, so we can try the usual trick of creating an anti-symmetric tensor by subtracting the same quantity with the indices swapped. That is, we define

$$F^{ij}=\partial^{i}A^{j}-\partial^{j}A^{i} \tag{3}$$

Now if we set $i=t$, $j=x$ we get

$$E_{x}=\partial^{t}A^{x}-\partial^{x}A^{t} \tag{4}$$

The raised partial derivative $\partial^{t}=-\partial_{t}$ while the other partials are the same in upper or lower form: $\partial^{x}=\partial_{x}$ and so on. Therefore, this equation is equivalent to

$$E_{x}=-\frac{\partial A^{x}}{\partial t}-\frac{\partial A^{t}}{\partial x} \tag{5}$$

If we're dealing with electrostatics, then all time derivatives are zero, so we get

$$E_x = -\frac{\partial A^t}{\partial x} \tag{6}$$

Comparing this with 2, we see that $A^t = \phi$.

What about the other components of A^i? If we take $i = x$, $j = y$ we get

$$B_z = \frac{\partial A^y}{\partial x} - \frac{\partial A^x}{\partial y} \tag{7}$$

The RHS is the z component of $\nabla \times \mathbf{A}$, and we get the other 2 components of the curl from the other 2 elements of F^{ij} involving B_y and B_x. Thus the spatial components of A^i form the magnetic vector potential.

Our first equation involving F^{ij} was the one relating it to the four-current:

$$\partial_j F^{ij} = 4\pi k J^i \tag{8}$$

Writing this in terms of the four-potential:

$$\partial_j \partial^i A^j - \partial_j \partial^j A^i = 4\pi k J^i \tag{9}$$

In the same way as we did in classical electrodynamics, we can specify a gauge transformation for the four-potential. That is, we replace A^i by $A^i + \partial^i \Lambda$, where Λ is an arbitrary scalar function. Plugging this into 3 we get

$$\begin{aligned} F^{ij} &= \partial^i \left(A^j + \partial^j \Lambda\right) - \partial^j \left(A^i + \partial^i \Lambda\right) && (10) \\ &= \partial^i A^j - \partial^j A^i + \partial^i \partial^j \Lambda - \partial^j \partial^i \Lambda && (11) \\ &= \partial^i A^j - \partial^j A^i && (12) \end{aligned}$$

since the order of the partial derivatives in the second line doesn't matter. Thus this transformation leaves the EM tensor unaltered and since it is only the electric and magnetic fields that are measurable, it leaves the physics unchanged.

If we choose Λ such that $\partial_j A^j = 0$ (it turns out this is always possible; it's called the Lorenz gauge), then 9 becomes

$$-\partial_j \partial^j A^i = 4\pi k J^i \tag{13}$$

In the static case, the time derivative is zero and this equation gives us Poisson's equation for electrostatics (from the $i = t$ term) and for magnetostatics (from the other 3 terms). That is, for $i = t$

$$\nabla^2\phi = -4\pi k\rho \tag{14}$$

and for the other 3 components

$$\nabla^2\mathbf{A} = -4\pi k\mathbf{J} \tag{15}$$

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