ELECTROMAGNETIC FIELD TENSOR: JUSTIFICATION

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Required math: algebra

Required physics: special relativity

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 7; Problems 7.5.

We've been using the electromagnetic (EM) field tensor F^{ij} in several problems without saying where it comes from, so it's time to fill in the gap with a look at how the form of this tensor was deduced.

To say this is a derivation of the EM tensor is probably stretching things a bit; it's more of a plausibility argument. As always in relativity, the idea is to generalize the equations of classical physics by putting them in tensor form.

We start with the charge density and current. As we've already seen, densities (of either mass or charge) are not invariant under a Lorentz transformation because of length contraction. If we start with a charge density ρ at rest and transform to an inertial frame moving at velocity β, the density becomes $\gamma\rho$. However, we also generate a current due to the motion of the charge, which is $\rho\beta$.

The four-current $\mathbf{J}$ is defined as a four-vector whose t component is the charge density and whose spatial components are the current. That is, in a frame moving with velocity β relative to the charge, we have

$$\mathbf{J}=[\gamma\rho,\gamma\beta_x\rho,\gamma\beta_y\rho,\gamma\beta_z\rho] \tag{1}$$

In the charge's rest frame, this is

$$\mathbf{J}=[\rho,0,0,0] \tag{2}$$

so the invariant square of the four-current is

$$J^2=-\rho^2 \tag{3}$$

The next step is to look at Gauss's law in differential form. This has the form

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0} \tag{4}$$

Moore uses the alternative form

$$\nabla\cdot\mathbf{E}=4\pi k\rho \tag{5}$$

where $k=1/4\pi\epsilon_0$. If we generalize the RHS to the four-current (which is a four-vector), then we need to make the LHS a four-vector as well. In its current form, $\nabla\cdot\mathbf{E}$ is a scalar, so we need to find some four-vector of which this is the 0 component (since ρ is the 0 component of $\mathbf{J}$). Since we need to take the derivative on the LHS, we can try the form

$$\partial_j F^{ij}=4\pi kJ^i \tag{6}$$

where F^{ij} is a rank-2 tensor to be determined.

Note that there is one slight snag in this argument. In general, the derivative of a tensor is *not* another tensor, so the quantity on the LHS is not a tensor in a general coordinate system. However, if the transformation between coordinate systems involves partial derivatives that are constants (that is, $\partial x'^i/\partial x^j=$ constant), then the derivative of a tensor *is* a tensor. For Lorentz transformations, this condition is true, since the transformation depends only on the (constant) relative velocity of the inertial frames. Thus in special relativity, the above equation is valid.

To make this equation consistent with Gauss's law, therefore, we need

$$F^{ij}=\left[\begin{array}{cccc} - & E_x & E_y & E_z\\ - & - & - & -\\ - & - & - & -\\ - & - & - & - \end{array}\right] \tag{7}$$

where the dashed entries are to be determined.

Next we look at the electrostatic force, which in Newtonian terms is

$$\frac{d\mathbf{p}}{dt}=q\mathbf{E} \tag{8}$$

If we generalize the LHS to $dp^i/d\tau$, we again need the RHS to be a four-vector. Since F^{ij} is a rank-2 tensor, we need to contract it with something to give us a four-vector. Taking partial derivatives won't work here, since the force equation involves $\mathbf{E}$ on its own, not its derivatives. At this point, we make a leap of logic and propose that we contract F^{ij} with the four-velocity u_j. This certainly isn't a derivation; its only justification is that it works. So we get the relativistic form of the force law:

$$\frac{dp^i}{d\tau}=qF^{ij}u_j \tag{9}$$

Note that u_j is the covariant version of the four-velocity, so its time component $u_0 = -u^0$, while the other components are the same in both forms.

From here, we can use the following identity:

$$\begin{aligned}
\frac{d(\mathbf{p}\cdot\mathbf{p})}{d\tau} &= \frac{d\left(p_i p^i\right)}{d\tau} && (10)\\
&= \frac{d\left(\eta_{ij} p^j p^i\right)}{d\tau} && (11)\\
&= \eta_{ij}\left(\frac{dp^i}{d\tau}p^j + \frac{dp^j}{d\tau}p^i\right) && (12)\\
&= 2\frac{dp^i}{d\tau}\eta_{ij}p^j && (13)\\
&= 2p_i\frac{dp^i}{d\tau} && (14)
\end{aligned}$$

In line 3 we used the fact that in flat space η_{ij} is a constant. In line 4, we used the symmetry of the flat space metric: $\eta_{ij} = \eta_{ji}$ to swap the indices on the second term.

Now the invariant $\mathbf{p}\cdot\mathbf{p} = -m^2$ is a constant, so $\frac{d(\mathbf{p}\cdot\mathbf{p})}{d\tau} = 0$. Therefore

$$\begin{aligned}
2p_i\frac{dp^i}{d\tau} &= 2\left(mu_i\right)\left(qF^{ij}u_j\right) && (15)\\
&= 2qmu_i u_j F^{ij} && (16)\\
&= 0 && (17)
\end{aligned}$$

In order for this to be true in general, that is for all four-velocities, we need to impose a condition on F^{ij}. We *could* just require $F^{ij} = 0$, but that wouldn't get us anywhere, since that would mean the electric field would have to be zero. The trick lies in the summation; if we require $F^{ij} = -F^{ji}$ then

$$\begin{aligned}
u_i u_j F^{ij} &= -u_i u_j F^{ji} && (18)\\
&= -u_i u_j F^{ij} && (19)
\end{aligned}$$

where in the last line we swapped the two dummy indices. Thus if F^{ij} is anti-symmetric, the condition is automatically satisfied. This allows us to fill out the tensor a bit more:

$$F^{ij}=\left[\begin{array}{cccc}0 & E_x & E_y & E_z\\ -E_x & 0 & - & -\\ -E_y & - & 0 & -\\ -E_z & - & - & 0\end{array}\right] \tag{20}$$

For a particle at rest, $u_j=0$ for $j=1,2,3$ and $u_0=-1$, so for $i=1,2,3$ from above we have

$$\begin{aligned}\frac{dp^i}{d\tau} &= qF^{i0}u_0 &(21)\\ &= -qF^{i0} &(22)\end{aligned}$$

which is the same as 8, since for a particle at rest $t=\tau$. So far, so good. However, we can now use 9 to see what happens when the particle is not at rest. We will fill in the tensor with the (admittedly suggestive) symbols shown:

$$F^{ij}=\left[\begin{array}{cccc}0 & E_x & E_y & E_z\\ -E_x & 0 & B_z & -B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0\end{array}\right] \tag{23}$$

At this point, we don't know what the Bs are; we're just using them as placeholders in the tensor. Then from 9 we get for the x component

$$\begin{aligned}\frac{dp^x}{d\tau} &= qF^{xj}u_j &(24)\\ &= q\left(-E_xu_0+0+u_yB_z-u_zB_y\right) &(25)\\ &= q\left(-E_xu_0+[\mathbf{u}\times\mathbf{B}]_x\right) &(26)\end{aligned}$$

This looks a lot like the Lorentz (this guy gets around) force law. If we use $p^i=mu^i$, $u^0=-u_0=\gamma$ and $u^i=u_i=\gamma v^i$ for $i=1,2,3$ we get

$$m\frac{dv^x}{d\tau}=q\gamma\left(E_x+[\mathbf{v}\times\mathbf{B}]_x\right) \tag{27}$$

Finally, we note that

$$\begin{aligned}\frac{dv^i}{d\tau} &= \frac{dv^i}{dt}\frac{dt}{d\tau} &(28)\\ &= \frac{dv^i}{dt}\gamma &(29)\end{aligned}$$

The final form is therefore

$$m\frac{dv^{x}}{dt}=q\left(E_{x}+[\mathbf{v}\times\mathbf{B}]_{x}\right) \tag{30}$$

with similar forms for the y and z components. This really is the Lorentz force law, and as we've seen from the derivation, it is valid in relativity as well as Newtonian physics, since we used relativistic four-vectors throughout, and never made the approximation of small velocities.

In one sense, we can take this as a *prediction* of the magnetic field and of the Lorentz force law, since these came out of our generalization of the equations for electrostatics (without any reference to magnetic fields).

There is a lot of 'try it and see' in this derivation, but it seems that's the way a lot of physics works. It's not like mathematics where we specify a minimal set of axioms and then rigorously derive every other result from them.

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