ELECTROMAGNETIC FIELD TENSOR: INVARIANCE OF INNER PRODUCT

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 7; Problem 7.7.

We've worked out earlier the scalar quantity $F_{ij}F^{ij}=2\left(B^2-E^2\right)$. We can check this for the case of a Lorentz transformation in flat space-time. We found there that

$$\begin{aligned}
E_x' &= E_x && (1)\\
E_y' &= \gamma E_y-\gamma\beta B_z && (2)\\
E_z' &= \gamma E_z+\gamma\beta B_y && (3)\\
B_x' &= B_x && (4)\\
B_y' &= -\gamma\beta E_y+\gamma B_z && (5)\\
B_z' &= -\gamma\beta E_z-\gamma B_y && (6)
\end{aligned}$$

Calculating the invariant in the new system we get

$$\begin{aligned}
B'^2-E'^2 &= B_x'^2+B_y'^2+B_z'^2+E_x'^2+E_y'^2+E_z'^2 && (7)\\
&= B_x^2+\left(-\gamma\beta E_y+\gamma B_z\right)^2+\left(-\gamma\beta E_z-\gamma B_y\right)^2- && (8)\\
&\quad E_x^2-\left(\gamma E_y-\gamma\beta B_z\right)^2-\left(\gamma E_z+\gamma\beta B_y\right)^2 && (9)\\
&= B_x^2+\left(B_y^2+B_z^2\right)\gamma^2\left(1-\beta^2\right)-E_x^2-\left(E_y^2+E_z^2\right)\gamma^2\left(1-\beta^2\right) && (10)\\
&= B^2-E^2 && (11)
\end{aligned}$$

since $\gamma=1/\sqrt{1-\beta^2}$. All the cross terms involving the product of a component of E and one of B cancel out between lines 2/3 and 4.