GEODESICS: PATHS OF LONGEST PROPER TIME

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.1.

One of the central problems in general relativity is the determination of *geodesic curves*, or just *geodesics* for short. These are the curves that a free particle (that is, a particle upon which no force acts, where 'force' in this case excludes gravity, since the effects of gravity are felt entirely through the curvature of space-time) will follow in a curved space-time.

To analyze this problem, it is useful to start, as always, with flat space-time in special relativity. In this case, a free particle travels in a straight line with constant velocity. If we look at this particle in its rest frame and place it at spatial coordinates $x=y=z=0$, its world line (in 2-d) is the t axis. That is, the particle starts off at the origin and just moves up the t axis with a constant x coordinate of $x=0$.

Now suppose we consider another particle that is *not* free, that is, it *is* acted on by some forces and as a result its world line is not a straight line. Suppose, however, that both particles start off at event A, which is at the origin at $t=0$, and also both pass through the event B at $x=0$, $t=1$. The free particle's world line between these two events is a straight line, as we've said, while the non-free particle's world line will be some path that can wander around any way it likes (as long as its speed doesn't exceed 1), provided that it comes back to meet the first particle at event B.

An invariant quantity for each particle is the proper time (that is, the time as measured by a clock that moves with the particle) that elapses between A and B. In the case of the free particle, the proper time is also the time coordinate in the measurement frame since the particle is at rest, so the proper time interval $\Delta\tau$ is just the time interval $\Delta t=1$. For the other particle, its proper time will *not* be the same as Δt, since the particle has a non-zero speed for part or all of its world line.

Intuitively, you might think that the non-free particle's proper time would be greater than that of the free particle, since obviously the path length of its world line is larger. However, remember that the invariant interval in relativity is calculated using the metric η_{ij}, whose 00 component is -1, so intuition leads you astray here. In the non-free particle's frame, its velocity is zero, so $-ds^2=d\tau^2$. In the free particle's frame (assuming motion only in the x direction):

$$-ds^2 = dt^2 - dx^2 \tag{1}$$
$$= dt^2\left(1-\beta^2\right) \tag{2}$$

Thus the relation between the two time intervals is

$$d\tau = \sqrt{1-\beta^2}dt \tag{3}$$

That is, the proper time interval for the moving particle is *less* than the time interval measured in the free particle's frame. In fact, if we integrate this between the times t_0 and t_1 (the time interval as measured by the free particle at rest), we get

$$\Delta\tau = \int_{t_0}^{t_1}\sqrt{1-\beta^2}dt \tag{4}$$

The square root is always between 0 and 1, so the maximum possible value of this integral is if $\beta = 0$ for the entire world line; that is, the particle is at rest. In other words, the free particle has the *maximum* proper time of all possible world lines that connect two events.

If the space-time metric had $\eta_{00} = +1$ instead of -1, then the integral would be $\int_{t_0}^{t_1}\sqrt{1+\beta^2}dt$, and then the *minimum* proper time would occur when $\beta = 0$, which is what our intuition (falsely) told us was true. Thus the non-intuitive result is purely the result of the metric having a negative time component.

Don't confuse this result with the time dilation principle, since they are considering different cases. Time dilation is an effect that occurs when two different observers (one moving at a speed v relative to the other) measure the time interval between two events on the *same* world line. In the current case, we're looking at the difference in proper time between two *different* world lines, as analyzed in the *same* inertial frame. This is why the limits on the integral in 4 are the same for all world lines; we're using the same time coordinate t (the time in the free particle's rest frame) in every case.

In flat space, then, the geodesic curve between two events is the one having the largest proper time for an object moving along that curve. The problem is therefore how to find a curve that maximizes the proper time.

Readers familiar with classical mechanics may recognize this problem as one that can be solved using the calculus of variations. We won't go through the derivation here, but the idea is that if we can identify a function called the *Lagrangian* L, which depends on a set of *generalized coordinates* q_i and *generalized velocities* $\dot{q}_i$ (where the dot indicates a time derivative) then we can define the *action* as

$$S \equiv \int_{t_A}^{t_B} L dt \tag{5}$$

The idea is that we need to minimize the action to find the path actually travelled by the object in the given time interval, and we do that by varying the path it follows until we minimize S. It turns out that this leads to the set of differential equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{6}$$

The solutions of these equations give the coordinates and velocities as functions of time; that is, they give the trajectory followed by the object.

In classical mechanics, the Lagrangian is the difference between the kinetic and potential energies: $L = T - V$. In relativity, as we've seen above, the Lagrangian is the proper time interval. (If you haven't seen this before, you can take it as god-given at this point. The important thing to understand is that, to find the path actually followed by an object, we need to find something we can maximize or minimize (the action in classical mechanics, the proper time in relativity), and then write that quantity as an integral over some parameter that describes the path followed.)

We've already found the solution of the Lagrangian problem for a free particle in flat space: the particle travels between the two events in a straight line at a constant velocity. The fundamental assumption that is made is that we can generalize this method of solution to curved space-time.

First, we use the fact that the proper time interval $\Delta\tau$ between two events A and B is given by integrating the invariant interval over the world line of the object. That is

$$\Delta\tau = \int_A^B \sqrt{-ds^2} \tag{7}$$

The minus sign occurs because $ds^2 = dx^2 + dy^2 + dz^2 - dt^2$, so in the rest frame of the object $ds^2 = -d\tau^2$.

Now suppose we have a general world path (we'll call it a 'path' rather than a 'line' since in general it need not be a straight line) that is described by a parameter σ that ranges from 0 at event A to 1 at event B. That is, all four coordinates on the world path are functions of σ: $x^i = x^i(\sigma)$. Then in a general curved space-time, we can say

$$ds^2 = g_{ij}dx^i dx^j \tag{8}$$

$$= g_{ij}\left(\frac{dx^i}{d\sigma}d\sigma\right)\left(\frac{dx^j}{d\sigma}d\sigma\right) \tag{9}$$

$$= g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}d\sigma^2 \tag{10}$$

Then for a given world path, the proper time that elapses along that path as we move from A to B is

$$\Delta\tau = \int_0^1 \sqrt{-g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}d\sigma \tag{11}$$

Remember that both the metric and the coordinates are, in general, functions of σ, so they can all vary as we move along the world path. This equation is in precisely the right form for a Lagrangian treatment. The proper time interval $\Delta\tau$ plays the role of the action S above, and the square root term becomes the Lagrangian L. The parameter σ plays the role of t in the classical equation.

Before we continue the analysis by plugging this form of the Lagrangian into 6 to get a differential equation involving the metric and coordinates, it's worth looking at an example of a relativistic system that can be analyzed without the full mathematical toolbox.

Consider an experimental setup in which we fire a laser downwards from the top of a tower of height z on the Earth's surface. One of the fundamentals of general relativity is the *principle of equivalence*, which states that a frame at rest in a gravitational field is equivalent to an accelerating frame in flat space. One way of visualizing this is to think of what it would feel like to be in an elevator in free space. As the elevator accelerates, you will feel a force pressing you into the floor in the same way you feel a force pressing you onto the ground if you stand still on the Earth's surface.

Applying this principle to our experiment, we see that in the time the light takes to reach the ground, the lab frame will have 'accelerated' from rest to a speed $\beta = gz$, where g is the acceleration of gravity. (Remember we're using $c = 1$, so the time t taken to reach the ground is $z/c = z$.) That is, the observer on the ground is effectively moving towards the light source with speed β so the light will appear blue-shifted, with the relation between the emitted wavelength λ_e and observed wavelength λ being

$$\frac{\lambda}{\lambda_e} = \sqrt{\frac{1-\beta}{1+\beta}} \tag{12}$$

(For a blue-shift, we must have $\lambda < \lambda_e$.)

If $\beta \ll 1$, we can approximate this by

$$\sqrt{\frac{1-\beta}{1+\beta}} \approx \left(1-\frac{1}{2}\beta\right)\left(1+\left(-\frac{1}{2}\right)(\beta)\right) \tag{13}$$

$$= 1-\beta+\mathcal{O}\left(\beta^2\right) \tag{14}$$

$$\approx 1-gz \tag{15}$$

Therefore

$$1-\frac{\lambda}{\lambda_e} \approx gz \tag{16}$$

$$\frac{\lambda_e-\lambda}{\lambda_e} \approx gz \tag{17}$$

The ground observer measures the time interval $dt = \lambda/c = \lambda$ between successive crests of the light wave. Since this is the same light wave as that emitted by the laser, the time interval between successive crests as measured at the laser must be $dT = \lambda_e = \frac{\lambda_e}{\lambda}\lambda = \frac{\lambda_e}{\lambda}dt \approx \frac{1}{1-gz}dt \approx (1+gz)\,dt$.

Now suppose that we fire a clock upwards from the ground, and that at a height of z it has a speed v. Due to time dilation, the proper time $d\tau$ between two infinitesimally close events measured by this moving clock will be shorter than the interval dT measured by a clock at rest at height z according to $d\tau = \sqrt{1-v^2}dT$. (Remember that moving clocks run slow, so the elapsed time on the moving clock is less than on the laser's clock.) Therefore, the proper time for the clock that travels from the ground up to the laser is related to the time measured by the clock on the ground by

$$d\tau = \sqrt{1-v^2}dT \tag{18}$$

$$\approx \sqrt{1-v^2}\,(1+gz)\,dt \tag{19}$$

$$\approx \left(1-\frac{1}{2}v^2\right)(1+gz)\,dt \tag{20}$$

$$\approx \left(1-\frac{1}{2}v^2+gz\right)dt \tag{21}$$

This gives us the Lagrangian as

$$\begin{aligned} L &= 1-\frac{1}{2}v^{2}+gz \qquad (22)\\ &= 1-\frac{1}{2}\left(\dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}\right)+gz \qquad (23)\end{aligned}$$

We can now write out the Lagrangian differential equations from 6:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial\dot{x}}-\frac{\partial L}{\partial x} &= -\ddot{x}=0 \qquad (24)\\ \frac{d}{dt}\frac{\partial L}{\partial\dot{y}}-\frac{\partial L}{\partial y} &= -\ddot{y}=0 \qquad (25)\\ \frac{d}{dt}\frac{\partial L}{\partial\dot{z}}-\frac{\partial L}{\partial z} &= -\ddot{z}+g=0 \qquad (26)\end{aligned}$$

The solutions are

$$\begin{aligned} x(t) &= x_{1}t+x_{0} \qquad (27)\\ y(t) &= y_{1}t+y_{0} \qquad (28)\\ z(t) &= \frac{1}{2}gt^{2}+z_{1}t+z_{0} \qquad (29)\end{aligned}$$

where x_0, x_1 and so on are constants of integration. The motion is therefore the standard parabolic trajectory, with constant speed in the x and y directions and the parabolic motion in the z direction. For a gravitational field pulling downwards, $g=-9.8\text{m sec}^{-2}$ at the Earth's surface, so the moving clock slows down as it rises.

PINGBACKS

Pingback: Geodesic equation: geodesics on a sphere
Pingback: Event horizon: time and space swap round
Pingback: Newtonian tidal effect
Pingback: Stress-energy tensor