

GEODESIC EQUATION: 2-D POLAR COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.3.

Another example of using the geodesic equation to calculate geodesics, this time in flat, 2-d space. We know that geodesics here are straight lines, but let's prove this using polar coordinates.

First, consider some arbitrary straight line. Draw a perpendicular from the origin to the line and call this distance b . The polar angle from the x axis to this perpendicular we define as α . The point where the perpendicular intersects the line marks the zero point for path length, so $s = 0$ there.

Starting with the perpendicular, we increase θ and draw the radius vector from the origin to the line at this angle. This vector has length r and intersects the line a distance s along from b . The vector, the perpendicular, and the line segment of length s make a right-angled triangle, so $r^2 = s^2 + b^2$, and the angle between the perpendicular and the vector is $\arctan \frac{s}{b}$, making the polar angle between the x axis and the vector $\theta = \alpha + \arctan \frac{s}{b}$. The parametric equations for the straight line in polar coordinates are then

$$(1) \quad r^2 = s^2 + b^2$$

$$(2) \quad \theta = \alpha + \arctan \frac{s}{b}$$

We now need to show that the geodesics given by the geodesic equation have the same form. The geodesic equation is

$$(3) \quad \frac{d}{d\tau} \left(g_{aj} \frac{dx^j}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0$$

The metric tensor in polar coordinates is

$$(4) \quad g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

The geodesic equation gives us

$$(5) \quad \frac{d^2 r}{ds^2} - r \left(\frac{d\theta}{ds} \right)^2 = 0$$

$$(6) \quad \frac{d}{ds} \left(r^2 \frac{d\theta}{ds} \right) = 0$$

The second equation can be integrated once to give

$$(7) \quad r^2 \frac{d\theta}{ds} = k$$

$$(8) \quad \frac{d\theta}{ds} = \frac{k}{r^2}$$

for some constant k . Substituting this into the first equation gives

$$(9) \quad \frac{d^2 r}{ds^2} = r \left(\frac{d\theta}{ds} \right)^2$$

$$(10) \quad = \frac{k^2}{r^3}$$

Using the condition

$$(11) \quad g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \left(\frac{ds}{ds} \right)^2 = +1$$

we get

$$(12) \quad \left(\frac{dr}{ds} \right)^2 + r^2 \left(\frac{d\theta}{ds} \right)^2 = 1$$

$$(13) \quad \frac{dr}{ds} = \pm \sqrt{1 - \frac{k^2}{r^2}}$$

This is the first integral of the second derivative above, as we can verify:

$$(14) \quad \frac{d^2 r}{ds^2} = \pm \frac{1}{2} \left(1 - \frac{k^2}{r^2} \right)^{-1/2} \left(\frac{2k^2}{r^3} \right) \frac{dr}{ds}$$

$$(15) \quad = \frac{k^2}{r^3}$$

We can now integrate the first derivative to get

$$(16) \quad \pm \int \left(1 - \frac{k^2}{r^2}\right)^{-1/2} dr = \int ds$$

$$(17) \quad \pm \sqrt{r^2 - k^2} = s + s_0$$

If we take the constant $k = b$ and $s_0 = 0$, then $r^2 = s^2 + b^2$ which agrees with the radial equation for the straight line above. If we then substitute this into $d\theta/ds$, we get

$$(18) \quad \frac{d\theta}{ds} = \frac{k}{r^2}$$

$$(19) \quad \theta = \int \frac{k}{k^2 + s^2} ds$$

$$(20) \quad = \arctan \frac{s}{k} + \alpha$$

where we can identify the constant of integration with the angle α defined above. This matches the θ equation for the straight line above. Thus the geodesic equation does indeed generate straight lines for the geodesics.