

## GEODESIC EQUATION: 2-D POLAR COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.3.

Another example of using the geodesic equation to calculate geodesics, this time in flat, 2-d space. We know that geodesics here are straight lines, but let's prove this using polar coordinates.

First, consider some arbitrary straight line. Draw a perpendicular from the origin to the line and call this distance  $b$ . The polar angle from the  $x$  axis to this perpendicular we define as  $\alpha$ . The point where the perpendicular intersects the line marks the zero point for path length, so  $s = 0$  there.

Starting with the perpendicular, we increase  $\theta$  and draw the radius vector from the origin to the line at this angle. This vector has length  $r$  and intersects the line a distance  $s$  along from  $b$ . The vector, the perpendicular, and the line segment of length  $s$  make a right-angled triangle, so  $r^2 = s^2 + b^2$ , and the angle between the perpendicular and the vector is  $\arctan \frac{s}{b}$ , making the polar angle between the  $x$  axis and the vector  $\theta = \alpha + \arctan \frac{s}{b}$ . The parametric equations for the straight line in polar coordinates are then

$$r^2 = s^2 + b^2 \quad (1)$$

$$\theta = \alpha + \arctan \frac{s}{b} \quad (2)$$

We now need to show that the geodesics given by the geodesic equation have the same form. The geodesic equation is

$$\frac{d}{d\tau} \left( g_{aj} \frac{dx^j}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0 \quad (3)$$

The metric tensor in polar coordinates is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \quad (4)$$

The geodesic equation gives us

$$\frac{d^2r}{ds^2} - r \left( \frac{d\theta}{ds} \right)^2 = 0 \quad (5)$$

$$\frac{d}{ds} \left( r^2 \frac{d\theta}{ds} \right) = 0 \quad (6)$$

The second equation can be integrated once to give

$$r^2 \frac{d\theta}{ds} = k \quad (7)$$

$$\frac{d\theta}{ds} = \frac{k}{r^2} \quad (8)$$

for some constant  $k$ . Substituting this into the first equation gives

$$\frac{d^2r}{ds^2} = r \left( \frac{d\theta}{ds} \right)^2 \quad (9)$$

$$= \frac{k^2}{r^3} \quad (10)$$

Using the condition

$$g^{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \left( \frac{ds}{ds} \right)^2 = +1 \quad (11)$$

we get

$$\left( \frac{dr}{ds} \right)^2 + r^2 \left( \frac{d\theta}{ds} \right)^2 = 1 \quad (12)$$

$$\frac{dr}{ds} = \pm \sqrt{1 - \frac{k^2}{r^2}} \quad (13)$$

This is the first integral of the second derivative above, as we can verify:

$$\frac{d^2r}{ds^2} = \pm \frac{1}{2} \left( 1 - \frac{k^2}{r^2} \right)^{-1/2} \left( 2 \frac{k^2}{r^3} \right) \frac{dr}{ds} \quad (14)$$

$$= \frac{k^2}{r^3} \quad (15)$$

We can now integrate the first derivative to get

$$\pm \int \left(1 - \frac{k^2}{r^2}\right)^{-1/2} dr = \int ds \quad (16)$$

$$\pm \sqrt{r^2 - k^2} = s + s_0 \quad (17)$$

If we take the constant  $k = b$  and  $s_0 = 0$ , then  $r^2 = s^2 + b^2$  which agrees with the radial equation for the straight line above. If we then substitute this into  $d\theta/ds$ , we get

$$\frac{d\theta}{ds} = \frac{k}{r^2} \quad (18)$$

$$\theta = \int \frac{k}{k^2 + s^2} ds \quad (19)$$

$$= \arctan \frac{s}{k} + \alpha \quad (20)$$

where we can identify the constant of integration with the angle  $\alpha$  defined above. This matches the  $\theta$  equation for the straight line above. Thus the geodesic equation does indeed generate straight lines for the geodesics.