GEODESIC EQUATION: 2-D POLAR COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.3.

Another example of using the geodesic equation to calculate geodesics, this time in flat, 2-d space. We know that geodesics here are straight lines, but let's prove this using polar coordinates.

First, consider some arbitrary straight line. Draw a perpendicular from the origin to the line and call this distance b. The polar angle from the x axis to this perpendicular we define as α. The point where the perpendicular intersects the line marks the zero point for path length, so $s=0$ there.

Starting with the perpendicular, we increase θ and draw the radius vector from the origin to the line at this angle. This vector has length r and intersects the line a distance s along from b. The vector, the perpendicular, and the line segment of length s make a right-angled triangle, so $r^2=s^2+b^2$, and the angle between the perpendicular and the vector is $\arctan\frac{s}{b}$, making the polar angle between the x axis and the vector $\theta=\alpha+\arctan\frac{s}{b}$. The parametric equations for the straight line in polar coordinates are then

$$\begin{aligned} r^2 &= s^2+b^2 & (1)\\ \theta &= \alpha+\arctan\frac{s}{b} & (2) \end{aligned}$$

We now need to show that the geodesics given by the geodesic equation have the same form. The geodesic equation is

$$\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right)-\frac{1}{2}\frac{\partial g_{ij}}{\partial x^a}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0 \tag{3}$$

The metric tensor in polar coordinates is

$$g_{ij}=\left[\begin{array}{cc} 1 & 0\\ 0 & r^2 \end{array}\right] \tag{4}$$

The geodesic equation gives us

$$\frac{d^2r}{ds^2} - r\left(\frac{d\theta}{ds}\right)^2 = 0 \quad (5)$$

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = 0 \quad (6)$$

The second equation can be integrated once to give

$$r^2\frac{d\theta}{ds} = k \quad (7)$$

$$\frac{d\theta}{ds} = \frac{k}{r^2} \quad (8)$$

for some constant k. Substituting this into the first equation gives

$$\frac{d^2r}{ds^2} = r\left(\frac{d\theta}{ds}\right)^2 \quad (9)$$

$$= \frac{k^2}{r^3} \quad (10)$$

Using the condition

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \left(\frac{ds}{ds}\right)^2 = +1 \quad (11)$$

we get

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = 1 \quad (12)$$

$$\frac{dr}{ds} = \pm\sqrt{1-\frac{k^2}{r^2}} \quad (13)$$

This is the first integral of the second derivative above, as we can verify:

$$\frac{d^2r}{ds^2} = \pm\frac{1}{2}\left(1-\frac{k^2}{r^2}\right)^{-1/2}\left(2\frac{k^2}{r^3}\right)\frac{dr}{ds} \quad (14)$$

$$= \frac{k^2}{r^3} \quad (15)$$

We can now integrate the first derivative to get

$$\pm\int\left(1-\frac{k^2}{r^2}\right)^{-1/2}dr = \int ds \tag{16}$$

$$\pm\sqrt{r^2-k^2} = s+s_0 \tag{17}$$

If we take the constant $k=b$ and $s_0=0$, then $r^2=s^2+b^2$ which agrees with the radial equation for the straight line above. If we then substitute this into $d\theta/ds$, we get

$$\frac{d\theta}{ds} = \frac{k}{r^2} \tag{18}$$

$$\theta = \int\frac{k}{k^2+s^2}ds \tag{19}$$

$$= \arctan\frac{s}{k}+\alpha \tag{20}$$

where we can identify the constant of integration with the angle α defined above. This matches the θ equation for the straight line above. Thus the geodesic equation does indeed generate straight lines for the geodesics.