

## GEODESIC EQUATION AND FOUR-VELOCITY

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.4.

The geodesic equation is

$$(0.1) \quad \frac{d}{d\tau} \left( g_{aj} \frac{dx^j}{d\tau} \right) - \frac{1}{2} \partial_a g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0$$

We can write this in terms of the four-velocity  $u^i \equiv dx^i/d\tau$  if we expand the first derivative:

$$(0.2) \quad \frac{d}{d\tau} \left( g_{aj} \frac{dx^j}{d\tau} \right) = \partial_i g_{aj} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} + g_{aj} \frac{d}{d\tau} \left( \frac{dx^j}{d\tau} \right)$$

$$(0.3) \quad = \partial_i g_{aj} u^i u^j + g_{aj} \frac{du^j}{d\tau}$$

Therefore, the geodesic equation is

$$(0.4) \quad \partial_i g_{aj} u^i u^j + g_{aj} \frac{du^j}{d\tau} - \frac{1}{2} \partial_a g_{ij} u^i u^j = 0$$

We can express this in a different form by multiplying by  $u^a$  and summing:

$$(0.5) \quad u^a \partial_i g_{aj} u^i u^j + g_{aj} u^a \frac{du^j}{d\tau} - \frac{1}{2} u^a \partial_a g_{ij} u^i u^j = g_{aj} u^a \frac{du^j}{d\tau} + \frac{1}{2} u^a u^i u^j \partial_a g_{ij}$$

$$(0.6) \quad = g_{aj} u^a \frac{du^j}{d\tau} + \frac{1}{2} u^i u^j \frac{dg_{ij}}{d\tau}$$

$$(0.7) \quad = \frac{1}{2} \frac{d}{d\tau} (g_{ij} u^i u^j)$$

$$(0.8) \quad = \frac{1}{2} \frac{d}{d\tau} (\mathbf{u} \cdot \mathbf{u})$$

The geodesic equation thus confirms that  $\frac{d}{d\tau} (\mathbf{u} \cdot \mathbf{u}) = 0$  along a geodesic, which we knew beforehand, since  $\mathbf{u} \cdot \mathbf{u} = -1$  is an invariant.

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