GEODESIC EQUATION AND FOUR-VELOCITY

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.4.

The geodesic equation is

$$\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right)-\frac{1}{2}\partial_a g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0 \tag{1}$$

We can write this in terms of the four-velocity $u^i\equiv dx^i/d\tau$ if we expand the first derivative:

$$\begin{aligned}
\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right) &= \partial_i g_{aj}\frac{\partial x^i}{\partial\tau}\frac{\partial x^j}{\partial\tau}+g_{aj}\frac{d}{d\tau}\left(\frac{dx^j}{d\tau}\right) && (2)\\
&= \partial_i g_{aj}u^i u^j+g_{aj}\frac{du^j}{d\tau} && (3)
\end{aligned}$$

Therefore, the geodesic equation is

$$\partial_i g_{aj}u^i u^j+g_{aj}\frac{du^j}{d\tau}-\frac{1}{2}\partial_a g_{ij}u^i u^j=0 \tag{4}$$

We can express this in a different form by multiplying by u^a and summing:

$$\begin{aligned}
u^a\partial_i g_{aj}u^i u^j+g_{aj}u^a\frac{du^j}{d\tau}-\frac{1}{2}u^a\partial_a g_{ij}u^i u^j &= g_{aj}u^a\frac{du^j}{d\tau}+\frac{1}{2}u^a u^i u^j\partial_a g_{ij} && (5)\\
&= g_{aj}u^a\frac{du^j}{d\tau}+\frac{1}{2}u^i u^j\frac{dg_{ij}}{d\tau} && (6)\\
&= \frac{1}{2}\frac{d}{d\tau}\left(g_{ij}u^i u^j\right) && (7)\\
&= \frac{1}{2}\frac{d}{d\tau}\left(\mathbf{u}\cdot\mathbf{u}\right) && (8)
\end{aligned}$$

The geodesic equation thus confirms that $\frac{d}{d\tau}(\mathbf{u}\cdot\mathbf{u})=0$ along a geodesic, which we knew beforehand, since $\mathbf{u}\cdot\mathbf{u}=-1$ is an invariant.

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