

GEODESIC EQUATION: 2-D SPACE-TIME

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.5.

This is an example of the geodesic equation in a 2-d space-time (with one time and one space dimension). The metric is given in a general way as

$$(0.1) \quad ds^2 = -dt^2 + f^2(q) dq^2$$

where q is the generalized spatial coordinate, and f is an arbitrary function. The metric tensor is then

$$(0.2) \quad g_{ij} = \begin{bmatrix} -1 & 0 \\ 0 & f^2(q) \end{bmatrix}$$

Using the time component of the geodesic equation, we set $a = t$ in:

$$(0.3) \quad \frac{d}{d\tau} \left(g_{aj} \frac{dx^j}{d\tau} \right) - \frac{1}{2} \partial_a g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0$$

We get

$$(0.4) \quad -\frac{d^2t}{d\tau^2} = 0$$

From this we conclude that

$$(0.5) \quad \frac{dt}{d\tau} = k$$

for some constant k .

Using the condition $\mathbf{u} \cdot \mathbf{u} = -1$ we get

$$(0.6) \quad g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = - \left(\frac{dt}{d\tau} \right)^2 + f^2 \left(\frac{dq}{d\tau} \right)^2$$

$$(0.7) \quad = -k^2 + f^2 \left(\frac{dq}{d\tau} \right)^2$$

$$(0.8) \quad = -1$$

$$(0.9) \quad \frac{dq}{d\tau} = \pm \frac{1}{f} \sqrt{(k^2 - 1)}$$

We can write this in terms of dq/dt :

$$(0.10) \quad \frac{dq}{d\tau} = \frac{dq}{dt} \frac{dt}{d\tau}$$

$$(0.11) \quad = k \frac{dq}{dt}$$

$$(0.12) \quad \frac{dq}{dt} = \pm \frac{1}{f(q)} \sqrt{\left(1 - \frac{1}{k^2}\right)}$$

That is, the geodesic is the solution of this differential equation.

If we define a new coordinate system in which $t' = t$ and $q' \equiv F(q)$ is the antiderivative (integral) of $f(q)$ then we can transform the metric tensor to this new coordinate system using the standard transformation formula

$$(0.13) \quad g'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} g_{kl}$$

By implicit differentiation:

$$(0.14) \quad \frac{\partial q'}{\partial q'} = \frac{\partial F}{\partial q} \frac{\partial q}{\partial q'} + \frac{\partial F}{\partial t} \frac{\partial t}{\partial q'}$$

$$(0.15) \quad 1 = f(q) \frac{\partial q}{\partial q'} + 0$$

$$(0.16) \quad \frac{\partial q}{\partial q'} = \frac{1}{f(q)}$$

The only other non-zero derivative is

$$(0.17) \quad \frac{\partial t}{\partial t'} = 1$$

The new metric is therefore

$$(0.18) \quad g'_{ij} = \begin{bmatrix} -1 \left(\frac{\partial t}{\partial t'} \right)^2 & 0 \\ 0 & f^2(q) \left(\frac{\partial q}{\partial q'} \right)^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is the metric of flat space-time in rectangular coordinates. Thus any metric with $g_{tt} = -1$ represents flat space-time, since $f(q)$ is just a transformation of the flat metric using different coordinates.