GEODESIC EQUATION: 2-D SPACE-TIME

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.5.

This is an example of the geodesic equation in a 2-d space-time (with one time and one space dimension). The metric is given in a general way as

$$ds^2 = -dt^2 + f^2(q)\,dq^2 \tag{1}$$

where q is the generalized spatial coordinate, and f is an arbitrary function. The metric tensor is then

$$g_{ij} = \begin{bmatrix} -1 & 0 \\ 0 & f^2(q) \end{bmatrix} \tag{2}$$

Using the time component of the geodesic equation, we set $a = t$ in:

$$\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right) - \frac{1}{2}\partial_a g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0 \tag{3}$$

We get

$$-\frac{d^2t}{d\tau^2} = 0 \tag{4}$$

From this we conclude that

$$\frac{dt}{d\tau} = k \tag{5}$$

for some constant k.

Using the condition $\mathbf{u}\cdot\mathbf{u} = -1$ we get

$$
\begin{aligned}
g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} &= -\left(\frac{dt}{d\tau}\right)^2 + f^2\left(\frac{dq}{d\tau}\right)^2 && (6)\\
&= -k^2 + f^2\left(\frac{dq}{d\tau}\right)^2 && (7)\\
&= -1 && (8)\\
\frac{dq}{d\tau} &= \pm\frac{1}{f}\sqrt{(k^2-1)} && (9)
\end{aligned}
$$

We can write this in terms of dq/dt:

$$
\begin{aligned}
\frac{dq}{d\tau} &= \frac{dq}{dt}\frac{dt}{d\tau} && (10)\\
&= k\frac{dq}{dt} && (11)\\
\frac{dq}{dt} &= \pm\frac{1}{f(q)}\sqrt{\left(1-\frac{1}{k^2}\right)} && (12)
\end{aligned}
$$

That is, the geodesic is the solution of this differential equation.

If we define a new coordinate system in which $t' = t$ and $q' \equiv F(q)$ is the antiderivative (integral) of $f(q)$ then we can transform the metric tensor to this new coordinate system using the standard transformation formula

$$
g'_{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}g_{kl} \tag{13}
$$

By implicit differentiation:

$$
\begin{aligned}
\frac{\partial q'}{\partial q'} &= \frac{\partial F}{\partial q}\frac{\partial q}{\partial q'} + \frac{\partial F}{\partial t}\frac{\partial t}{\partial q'} && (14)\\
1 &= f(q)\frac{\partial q}{\partial q'} + 0 && (15)\\
\frac{\partial q}{\partial q'} &= \frac{1}{f(q)} && (16)
\end{aligned}
$$

The only other non-zero derivative is

$$
\frac{\partial t}{\partial t'} = 1 \tag{17}
$$

The new metric is therefore

$$g'_{ij} = \begin{bmatrix} -1\left(\frac{\partial t}{\partial t'}\right)^2 & 0 \\ 0 & f^2(q)\left(\frac{\partial q}{\partial q'}\right)^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{18}$$

This is the metric of flat space-time in rectangular coordinates. Thus any metric with $g_{tt} = -1$ represents flat space-time, since $f(q)$ is just a transformation of the flat metric using different coordinates.