GEODESIC EQUATION IN 2-D: EXPONENTIAL METRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.6.

This is another example of the geodesic equation in a 2-d space-time (with one time and one space dimension). The metric in this case is

$$ds^2 = -e^{-x/a}dt^2 + dx^2 \tag{1}$$

The metric tensor is then

$$g_{ij} = \begin{bmatrix} -e^{-x/a} & 0 \\ 0 & 1 \end{bmatrix} \tag{2}$$

Using the time component of the geodesic equation, we get

$$\begin{aligned} \frac{d}{d\tau}\left(-e^{-x/a}\frac{dt}{d\tau}\right) &= 0 && (3)\\ \frac{1}{a}\frac{dx}{d\tau}e^{-x/a}\frac{dt}{d\tau} - e^{-x/a}\frac{d^2t}{d\tau^2} &= 0 && (4)\\ \frac{d^2t}{d\tau^2} &= \frac{1}{a}\frac{dx}{d\tau}\frac{dt}{d\tau} && (5) \end{aligned}$$

This might look impossible to solve since we don't know $dx/d\tau$, but if we try (for a constant c):

$$\begin{aligned} \frac{dt}{d\tau} &= ce^{x/a} && (6)\\ \frac{d^2t}{d\tau^2} &= \frac{c}{a}\frac{dx}{d\tau}e^{x/a} && (7)\\ &= \frac{1}{a}\frac{dx}{d\tau}\frac{dt}{d\tau} && (8) \end{aligned}$$

Using the usual trick of requiring $\mathbf{u}\cdot\mathbf{u} = -1$ we get

$$\begin{aligned} g_{ij}u^i j^j &= -e^{-x/a}\left(\frac{dt}{d\tau}\right)^2+\left(\frac{dx}{d\tau}\right)^2 && (9)\\ &= -e^{-x/a}c^2e^{2x/a}+\left(\frac{dx}{d\tau}\right)^2=-1 && (10)\\ \frac{dx}{d\tau} &= \pm\sqrt{c^2e^{x/a}-1} && (11)\end{aligned}$$

Using the chain rule, we get

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{dx}{dt}\frac{dt}{d\tau} && (12)\\ &= ce^{x/a}\frac{dx}{dt} && (13)\\ \frac{dx}{dt} &= \pm\frac{\sqrt{c^2e^{x/a}-1}}{ce^{x/a}} && (14)\\ \pm\int\frac{ce^{x/a}}{\sqrt{c^2e^{x/a}-1}}dx &= t+t_0 && (15)\\ \pm\frac{2a}{c}\sqrt{c^2e^{x/a}-1}+\alpha &= t+t_0 && (16)\end{aligned}$$

where α and t_0 are constants of integration. If we set $t_0=0$ and consider x to be increasing from $x(0)$, then we get, by inverting the above equation:

$$x(t)=a\ln\left(\frac{1}{c^2}+\frac{(t-\alpha)^2}{4a^2}\right) \qquad (17)$$

At $t=0$, the position is x_0:

$$x_0=a\ln\left(\frac{1}{c^2}+\frac{\alpha^2}{4a^2}\right) \qquad (18)$$

From above, the initial velocity $\frac{dx}{d\tau}(0)\equiv u_0$ is then

$$\begin{aligned} u_0 &= \sqrt{c^2e^{x_0/a}-1} && (19)\\ &= \pm\frac{c\alpha}{2a} && (20)\end{aligned}$$

We can replace α by u_0 in the expression for $x(t)$ to get

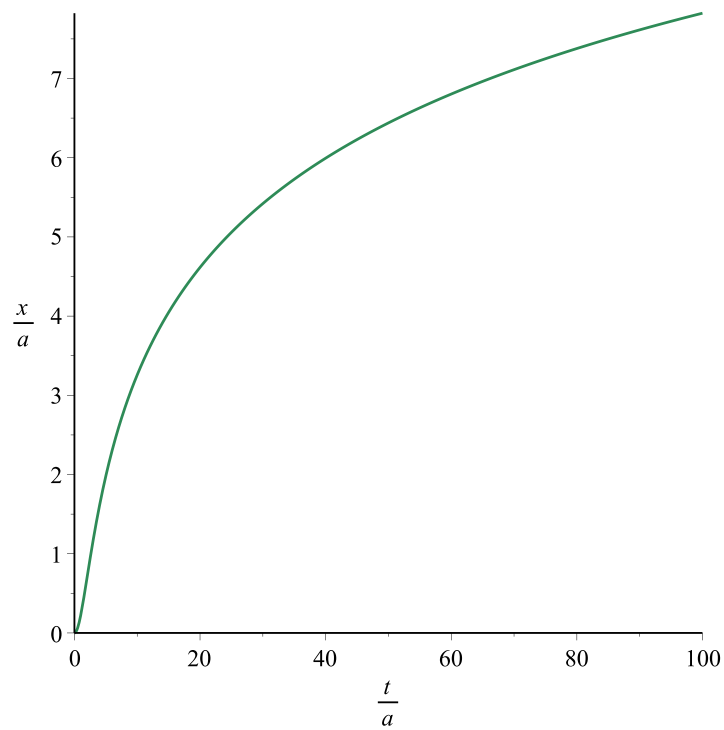
$$x(t) = a\ln\left(\frac{1}{c^2}+\frac{\left(t\pm\frac{2au_0}{c}\right)^2}{4a^2}\right) \tag{21}$$

$$= a\ln\left(\frac{1}{c^2}+\left(\frac{t}{2a}\pm\frac{u_0}{c}\right)^2\right) \tag{22}$$

If we require $x_0=u_0=0$ then from above we must have $a\ln\left(1/c^2\right)=0$, which gives $c=\pm1$ (since a appears in denominators, it can't be zero). This in turn requires $\alpha=0$, so the geodesic curve is given by

$$x(t)=a\ln\left(1+\frac{t^2}{4a^2}\right) \tag{23}$$

If we plot $\frac{x}{a}$ versus $\frac{t}{a}$ we get the following:



We can also work out $x(\tau)$ and $t(\tau)$ using the above equations. First, look at x:

$$\frac{dx}{d\tau} = \pm\sqrt{c^2 e^{x/a}-1} \tag{24}$$

$$\pm\int\frac{dx}{\sqrt{c^2 e^{x/a}-1}} = \tau \tag{25}$$

$$\pm 2a\arctan\left(\sqrt{c^2 e^{x/a}-1}\right)+\beta = \tau \tag{26}$$

where β is a constant of integration. If we require $\tau_0 = 0$ at $t = 0$ with the initial conditions above $x_0 = u_0 = 0$) then using the expression for x_0 above, we find that $\beta = 0$. Therefore (taking $c = 1$)

$$x(\tau) = a\ln\left(1+\tan^2\frac{\tau}{2a}\right) \tag{27}$$

From this we have from 6 (again, taking $c = 1$):

$$\frac{dt}{d\tau} = e^{x/a} \tag{28}$$

$$= 1+\tan^2\frac{\tau}{2a} \tag{29}$$

$$= \sec^2\frac{\tau}{2a} \tag{30}$$

Integrating we get

$$t(\tau) = \int\sec^2\frac{\tau}{2a}d\tau \tag{31}$$

$$= 2a\tan\frac{\tau}{2a} \tag{32}$$

where we've chosen the constant of integration so that $t(0) = 0$. As $\tau \to \pi a$, $t \to \infty$ so

$$\tau_{max} = \pi a \tag{33}$$