

GEODESIC EQUATION IN 2-D: EXPONENTIAL METRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.6.

This is another example of the geodesic equation in a 2-d space-time (with one time and one space dimension). The metric in this case is

$$ds^2 = -e^{-x/a} dt^2 + dx^2 \quad (1)$$

The metric tensor is then

$$g_{ij} = \begin{bmatrix} -e^{-x/a} & 0 \\ 0 & 1 \end{bmatrix} \quad (2)$$

Using the time component of the geodesic equation, we get

$$\frac{d}{d\tau} \left(-e^{-x/a} \frac{dt}{d\tau} \right) = 0 \quad (3)$$

$$\frac{1}{a} \frac{dx}{d\tau} e^{-x/a} \frac{dt}{d\tau} - e^{-x/a} \frac{d^2 t}{d\tau^2} = 0 \quad (4)$$

$$\frac{d^2 t}{d\tau^2} = \frac{1}{a} \frac{dx}{d\tau} \frac{dt}{d\tau} \quad (5)$$

This might look impossible to solve since we don't know $dx/d\tau$, but if we try (for a constant c):

$$\frac{dt}{d\tau} = ce^{x/a} \quad (6)$$

$$\frac{d^2 t}{d\tau^2} = \frac{c}{a} \frac{dx}{d\tau} e^{x/a} \quad (7)$$

$$= \frac{1}{a} \frac{dx}{d\tau} \frac{dt}{d\tau} \quad (8)$$

Using the usual trick of requiring $\mathbf{u} \cdot \mathbf{u} = -1$ we get

$$g_{ij}u^i u^j = -e^{-x/a} \left(\frac{dt}{d\tau} \right)^2 + \left(\frac{dx}{d\tau} \right)^2 \quad (9)$$

$$= -e^{-x/a} c^2 e^{2x/a} + \left(\frac{dx}{d\tau} \right)^2 = -1 \quad (10)$$

$$\frac{dx}{d\tau} = \pm \sqrt{c^2 e^{x/a} - 1} \quad (11)$$

Using the chain rule, we get

$$\frac{dx}{d\tau} = \frac{dx}{dt} \frac{dt}{d\tau} \quad (12)$$

$$= c e^{x/a} \frac{dx}{dt} \quad (13)$$

$$\frac{dx}{dt} = \pm \frac{\sqrt{c^2 e^{x/a} - 1}}{c e^{x/a}} \quad (14)$$

$$\pm \int \frac{c e^{x/a}}{\sqrt{c^2 e^{x/a} - 1}} dx = t + t_0 \quad (15)$$

$$\pm \frac{2a}{c} \sqrt{c^2 e^{x/a} - 1} + \alpha = t + t_0 \quad (16)$$

where α and t_0 are constants of integration. If we set $t_0 = 0$ and consider x to be increasing from $x(0)$, then we get, by inverting the above equation:

$$x(t) = a \ln \left(\frac{1}{c^2} + \frac{(t - \alpha)^2}{4a^2} \right) \quad (17)$$

At $t = 0$, the position is x_0 :

$$x_0 = a \ln \left(\frac{1}{c^2} + \frac{\alpha^2}{4a^2} \right) \quad (18)$$

From above, the initial velocity $\frac{dx}{d\tau}(0) \equiv u_0$ is then

$$u_0 = \sqrt{c^2 e^{x_0/a} - 1} \quad (19)$$

$$= \pm \frac{c\alpha}{2a} \quad (20)$$

We can replace α by u_0 in the expression for $x(t)$ to get

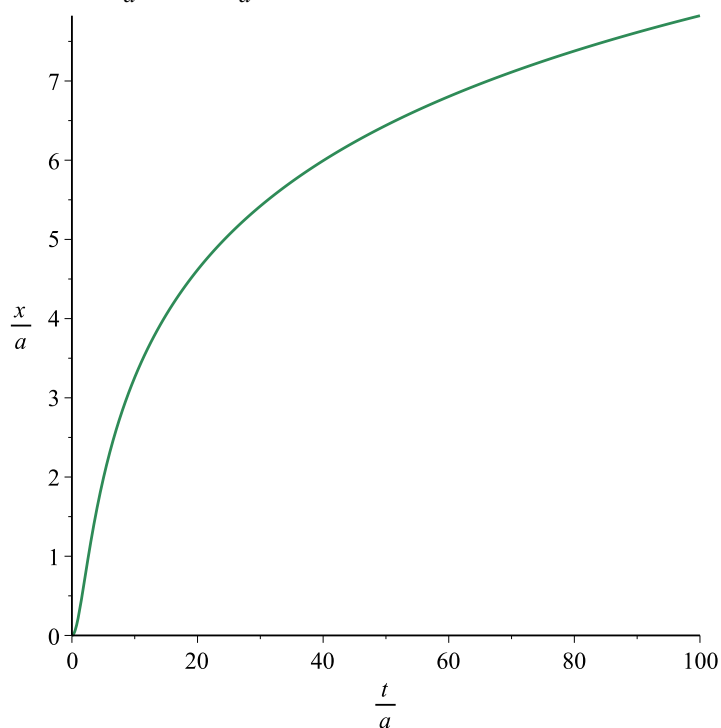
$$x(t) = a \ln \left(\frac{1}{c^2} + \frac{\left(t \pm \frac{2au_0}{c}\right)^2}{4a^2} \right) \quad (21)$$

$$= a \ln \left(\frac{1}{c^2} + \left(\frac{t}{2a} \pm \frac{u_0}{c}\right)^2 \right) \quad (22)$$

If we require $x_0 = u_0 = 0$ then from above we must have $a \ln(1/c^2) = 0$, which gives $c = \pm 1$ (since a appears in denominators, it can't be zero). This in turn requires $\alpha = 0$, so the geodesic curve is given by

$$x(t) = a \ln \left(1 + \frac{t^2}{4a^2} \right) \quad (23)$$

If we plot $\frac{x}{a}$ versus $\frac{t}{a}$ we get the following:



We can also work out $x(\tau)$ and $t(\tau)$ using the above equations. First, look at x :

$$\frac{dx}{d\tau} = \pm \sqrt{c^2 e^{x/a} - 1} \quad (24)$$

$$\pm \int \frac{dx}{\sqrt{c^2 e^{x/a} - 1}} = \tau \quad (25)$$

$$\pm 2a \arctan\left(\sqrt{c^2 e^{x/a} - 1}\right) + \beta = \tau \quad (26)$$

where β is a constant of integration. If we require $\tau_0 = 0$ at $t = 0$ with the initial conditions above $x_0 = u_0 = 0$) then using the expression for x_0 above, we find that $\beta = 0$. Therefore (taking $c = 1$)

$$x(\tau) = a \ln\left(1 + \tan^2 \frac{\tau}{2a}\right) \quad (27)$$

From this we have from 6 (again, taking $c = 1$):

$$\frac{dt}{d\tau} = e^{x/a} \quad (28)$$

$$= 1 + \tan^2 \frac{\tau}{2a} \quad (29)$$

$$= \sec^2 \frac{\tau}{2a} \quad (30)$$

Integrating we get

$$t(\tau) = \int \sec^2 \frac{\tau}{2a} d\tau \quad (31)$$

$$= 2a \tan \frac{\tau}{2a} \quad (32)$$

where we've chosen the constant of integration so that $t(0) = 0$. As $\tau \rightarrow \pi a$, $t \rightarrow \infty$ so

$$\tau_{max} = \pi a \quad (33)$$