

## GEODESIC EQUATION IN 2-D: EXPONENTIAL METRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.6.

This is another example of the geodesic equation in a 2-d space-time (with one time and one space dimension). The metric in this case is

$$(0.1) \quad ds^2 = -e^{-x/a} dt^2 + dx^2$$

The metric tensor is then

$$(0.2) \quad g_{ij} = \begin{bmatrix} -e^{-x/a} & 0 \\ 0 & 1 \end{bmatrix}$$

Using the time component of the geodesic equation, we get

$$(0.3) \quad \frac{d}{d\tau} \left( -e^{-x/a} \frac{dt}{d\tau} \right) = 0$$

$$(0.4) \quad \frac{1}{a} \frac{dx}{d\tau} e^{-x/a} \frac{dt}{d\tau} - e^{-x/a} \frac{d^2 t}{d\tau^2} = 0$$

$$(0.5) \quad \frac{d^2 t}{d\tau^2} = \frac{1}{a} \frac{dx}{d\tau} \frac{dt}{d\tau}$$

This might look impossible to solve since we don't know  $dx/d\tau$ , but if we try (for a constant  $c$ ):

$$(0.6) \quad \frac{dt}{d\tau} = ce^{x/a}$$

$$(0.7) \quad \frac{d^2 t}{d\tau^2} = \frac{c}{a} \frac{dx}{d\tau} e^{x/a}$$

$$(0.8) \quad = \frac{1}{a} \frac{dx}{d\tau} \frac{dt}{d\tau}$$

Using the usual trick of requiring  $\mathbf{u} \cdot \mathbf{u} = -1$  we get

$$(0.9) \quad g_{ij}u^i j^j = -e^{-x/a} \left( \frac{dt}{d\tau} \right)^2 + \left( \frac{dx}{d\tau} \right)^2$$

$$(0.10) \quad = -e^{-x/a} c^2 e^{2x/a} + \left( \frac{dx}{d\tau} \right)^2 = -1$$

$$(0.11) \quad \frac{dx}{d\tau} = \pm \sqrt{c^2 e^{x/a} - 1}$$

Using the chain rule, we get

$$(0.12) \quad \frac{dx}{d\tau} = \frac{dx}{dt} \frac{dt}{d\tau}$$

$$(0.13) \quad = c e^{x/a} \frac{dx}{dt}$$

$$(0.14) \quad \frac{dx}{dt} = \pm \frac{\sqrt{c^2 e^{x/a} - 1}}{c e^{x/a}}$$

$$(0.15) \quad \pm \int \frac{c e^{x/a}}{\sqrt{c^2 e^{x/a} - 1}} dx = t + t_0$$

$$(0.16) \quad \pm \frac{2a}{c} \sqrt{c^2 e^{x/a} - 1} + \alpha = t + t_0$$

where  $\alpha$  and  $t_0$  are constants of integration. If we set  $t_0 = 0$  and consider  $x$  to be increasing from  $x(0)$ , then we get, by inverting the above equation:

$$(0.17) \quad x(t) = a \ln \left( \frac{1}{c^2} + \frac{(t - \alpha)^2}{4a^2} \right)$$

At  $t = 0$ , the position is  $x_0$ :

$$(0.18) \quad x_0 = a \ln \left( \frac{1}{c^2} + \frac{\alpha^2}{4a^2} \right)$$

From above, the initial velocity  $\frac{dx}{d\tau}(0) \equiv u_0$  is then

$$(0.19) \quad u_0 = \sqrt{c^2 e^{x_0/a} - 1}$$

$$(0.20) \quad = \pm \frac{c\alpha}{2a}$$

We can replace  $\alpha$  by  $u_0$  in the expression for  $x(t)$  to get

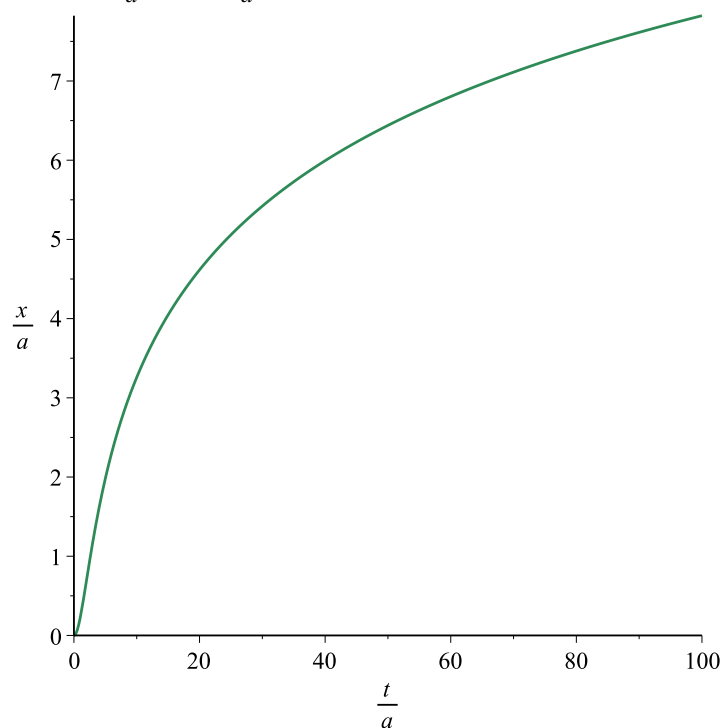
$$(0.21) \quad x(t) = a \ln \left( \frac{1}{c^2} + \frac{\left(t \pm \frac{2au_0}{c}\right)^2}{4a^2} \right)$$

$$(0.22) \quad = a \ln \left( \frac{1}{c^2} + \left(\frac{t}{2a} \pm \frac{u_0}{c}\right)^2 \right)$$

If we require  $x_0 = u_0 = 0$  then from above we must have  $a \ln(1/c^2) = 0$ , which gives  $c = \pm 1$  (since  $a$  appears in denominators, it can't be zero). This in turn requires  $\alpha = 0$ , so the geodesic curve is given by

$$(0.23) \quad x(t) = a \ln \left( 1 + \frac{t^2}{4a^2} \right)$$

If we plot  $\frac{x}{a}$  versus  $\frac{t}{a}$  we get the following:



We can also work out  $x(\tau)$  and  $t(\tau)$  using the above equations. First, look at  $x$ :

$$(0.24) \quad \frac{dx}{d\tau} = \pm \sqrt{c^2 e^{x/a} - 1}$$

$$(0.25) \quad \pm \int \frac{dx}{\sqrt{c^2 e^{x/a} - 1}} = \tau$$

$$(0.26) \quad \pm 2a \arctan\left(\sqrt{c^2 e^{x/a} - 1}\right) + \beta = \tau$$

where  $\beta$  is a constant of integration. If we require  $\tau_0 = 0$  at  $t = 0$  with the initial conditions above  $x_0 = u_0 = 0$ ) then using the expression for  $x_0$  above, we find that  $\beta = 0$ . Therefore (taking  $c = 1$ )

$$(0.27) \quad x(\tau) = a \ln\left(1 + \tan^2 \frac{\tau}{2a}\right)$$

From this we have from 0.6 (again, taking  $c = 1$ ):

$$(0.28) \quad \frac{dt}{d\tau} = e^{x/a}$$

$$(0.29) \quad = 1 + \tan^2 \frac{\tau}{2a}$$

$$(0.30) \quad = \sec^2 \frac{\tau}{2a}$$

Integrating we get

$$(0.31) \quad t(\tau) = \int \sec^2 \frac{\tau}{2a} d\tau$$

$$(0.32) \quad = 2a \tan \frac{\tau}{2a}$$

where we've chosen the constant of integration so that  $t(0) = 0$ . As  $\tau \rightarrow \pi a$ ,  $t \rightarrow \infty$  so

$$(0.33) \quad \tau_{max} = \pi a$$