

GEODESIC EQUATION: PARABOLOID

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.7.

This is an example of the geodesic equation in a 2-d curved space (with two space dimensions). We have a parabolic bowl with equation

$$(0.1) \quad z = br^2$$

where $r^2 = x^2 + y^2$ and b is a constant. We use the two coordinates r (as defined here) and the azimuthal angle ϕ . We've already looked at this example and found that the metric is

$$(0.2) \quad ds^2 = \left(1 + (2br)^2\right) dr^2 + r^2 d\phi^2$$

Using the geodesic equation for the two coordinates, we get

$$(0.3) \quad \frac{d}{ds} \left(\left(1 + (2br)^2\right) \frac{dr}{ds} \right) - \frac{1}{2} 8b^2 r \left(\frac{dr}{ds} \right)^2 - \frac{1}{2} 2r \left(\frac{d\phi}{ds} \right)^2 = 0$$

$$(0.4) \quad \frac{d}{ds} \left(r^2 \frac{d\phi}{ds} \right) = 0$$

The second equation can be integrated once to give

$$(0.5) \quad \frac{d\phi}{ds} = \frac{c}{r^2}$$

where c is a constant. Substituting this into the first equation and working out the first term, we get

$$(0.6) \quad 4b^2 r \left(\frac{dr}{ds} \right)^2 + \left(1 + (2br)^2\right) \frac{d^2 r}{ds^2} - \frac{c^2}{r^3} = 0$$

Integrating these equations directly isn't easy, but if we look at the special case of a curve defined by the condition $\phi = k$ for some constant k (that is,

a curve starting at the bottom of the bowl and going directly up the side of the bowl), then $d\phi = 0$ and from the metric:

$$(0.7) \quad \frac{dr}{ds} = \frac{1}{\sqrt{1 + (2br)^2}}$$

$$(0.8) \quad \frac{d^2r}{ds^2} = -\frac{4b^2r}{(1 + (2br)^2)^{3/2}} \frac{dr}{ds}$$

$$(0.9) \quad = -\frac{4b^2r}{(1 + (2br)^2)^2}$$

Inserting this into the above, we get

$$(0.10) \quad \frac{4b^2r}{1 + (2br)^2} - (1 + (2br)^2) \frac{4b^2r}{(1 + (2br)^2)^2} - \frac{c^2}{r^3} = -\frac{c^2}{r^3}$$

This will be equal to zero if $c = 0$, which also satisfies the condition $\frac{d\phi}{ds} = \frac{c}{r^2} = 0$. Thus these radial curves are in fact geodesics.