GEODESIC EQUATION: PARABOLOID

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.7.

This is an example of the geodesic equation in a 2-d curved space (with two space dimensions). We have a parabolic bowl with equation

$$z = br^2 \tag{1}$$

where $r^2 = x^2 + y^2$ and b is a constant. We use the two coordinates r (as defined here) and the azimuthal angle ϕ. We've already looked at this example and found that the metric is

$$ds^2 = \left(1 + (2br)^2\right) dr^2 + r^2 d\phi^2 \tag{2}$$

Using the geodesic equation for the two coordinates, we get

$$\frac{d}{ds}\left(\left(1+(2br)^2\right)\frac{dr}{ds}\right) - \frac{1}{2}8b^2 r\left(\frac{dr}{ds}\right)^2 - \frac{1}{2}2r\left(\frac{d\phi}{ds}\right)^2 = 0 \tag{3}$$

$$\frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0 \tag{4}$$

The second equation can be integrated once to give

$$\frac{d\phi}{ds} = \frac{c}{r^2} \tag{5}$$

where c is a constant. Substituting this into the first equation and working out the first term, we get

$$4b^2 r\left(\frac{dr}{ds}\right)^2 + \left(1+(2br)^2\right)\frac{d^2r}{ds^2} - \frac{c^2}{r^3} = 0 \tag{6}$$

Integrating these equations directly isn't easy, but if we look at the special case of a curve defined by the condition $\phi = k$ for some constant k (that is, a curve starting at the bottom of the bowl and going directly up the side of the bowl), then $d\phi = 0$ and from the metric:

$$\frac{dr}{ds} = \frac{1}{\sqrt{1+(2br)^2}} \tag{7}$$

$$\frac{d^2r}{ds^2} = -\frac{4b^2r}{\left(1+(2br)^2\right)^{3/2}}\frac{dr}{ds} \tag{8}$$

$$= -\frac{4b^2r}{\left(1+(2br)^2\right)^2} \tag{9}$$

Inserting this into the above, we get

$$\frac{4b^2r}{1+(2br)^2} - \left(1+(2br)^2\right)\frac{4b^2r}{\left(1+(2br)^2\right)^2} - \frac{c^2}{r^3} = -\frac{c^2}{r^3} \tag{10}$$

This will be equal to zero if $c = 0$, which also satisfies the condition $\frac{d\phi}{ds} = \frac{c}{r^2} = 0$. Thus these radial curves are in fact geodesics.