

## GEODESIC EQUATION: PARABOLOID

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 8; Problem 8.7.

This is an example of the geodesic equation in a 2-d curved space (with two space dimensions). We have a parabolic bowl with equation

$$z = br^2 \quad (1)$$

where  $r^2 = x^2 + y^2$  and  $b$  is a constant. We use the two coordinates  $r$  (as defined here) and the azimuthal angle  $\phi$ . We've already looked at this example and found that the metric is

$$ds^2 = \left(1 + (2br)^2\right) dr^2 + r^2 d\phi^2 \quad (2)$$

Using the geodesic equation for the two coordinates, we get

$$\frac{d}{ds} \left( \left(1 + (2br)^2\right) \frac{dr}{ds} \right) - \frac{1}{2} 8b^2 r \left( \frac{dr}{ds} \right)^2 - \frac{1}{2} 2r \left( \frac{d\phi}{ds} \right)^2 = 0 \quad (3)$$

$$\frac{d}{ds} \left( r^2 \frac{d\phi}{ds} \right) = 0 \quad (4)$$

The second equation can be integrated once to give

$$\frac{d\phi}{ds} = \frac{c}{r^2} \quad (5)$$

where  $c$  is a constant. Substituting this into the first equation and working out the first term, we get

$$4b^2 r \left( \frac{dr}{ds} \right)^2 + \left(1 + (2br)^2\right) \frac{d^2 r}{ds^2} - \frac{c^2}{r^3} = 0 \quad (6)$$

Integrating these equations directly isn't easy, but if we look at the special case of a curve defined by the condition  $\phi = k$  for some constant  $k$  (that is, a curve starting at the bottom of the bowl and going directly up the side of the bowl), then  $d\phi = 0$  and from the metric:

$$\frac{dr}{ds} = \frac{1}{\sqrt{1 + (2br)^2}} \quad (7)$$

$$\frac{d^2r}{ds^2} = -\frac{4b^2r}{\left(1 + (2br)^2\right)^{3/2}} \frac{dr}{ds} \quad (8)$$

$$= -\frac{4b^2r}{\left(1 + (2br)^2\right)^2} \quad (9)$$

Inserting this into the above, we get

$$\frac{4b^2r}{1 + (2br)^2} - \left(1 + (2br)^2\right) \frac{4b^2r}{\left(1 + (2br)^2\right)^2} - \frac{c^2}{r^3} = -\frac{c^2}{r^3} \quad (10)$$

This will be equal to zero if  $c = 0$ , which also satisfies the condition  $\frac{d\phi}{ds} = \frac{c}{r^2} = 0$ . Thus these radial curves are in fact geodesics.