SCHWARZSCHILD METRIC: GRAVITATIONAL REDSHIFT

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapters 9; Problem 9.1.

The relation between the proper time interval and Schwarzschild time interval for an object at rest is

$$\Delta\tau=\left(1-\frac{2GM}{r}\right)^{1/2}\Delta t \tag{0.1}$$

We can use this relation to derive a formula for the gravitational redshift. The key to this is that the wavelength of light is $\lambda=c\Delta\tau$, where $c=1$ is the speed of light and $\Delta\tau$ is the time interval as measured by an observer required for a single wavelength to be emitted or received. Why $\Delta\tau$ instead of Δt? As far as I understand it, this is because τ is the only correct measure of time for an object at rest. The Scwarzschild t coordinate, although it's called the 'time coordinate', isn't really a measure of time directly.

The redshift arises because if we emit a light beam at $r=r_E$ in a direction radially outwards from the mass M and receive the light beam at $r=r_R>r_E$, then the interval Δt required for the passage of a single wavelength must be the same at both the emitter and the receiver. Why? Because the metric doesn't depend on t.

The proper time interval, however, is not the same at the two points because of the relation above. Plugging in the values, we get

$$\frac{\lambda_R}{\lambda_E} = \frac{\Delta\tau_R}{\Delta\tau_E} \tag{0.2}$$

$$= \sqrt{\frac{1-2GM/r_R}{1-2GM/r_E}} \tag{0.3}$$

This formula could be used, for example, to calculate the redshift due to a star when observed from Earth.

If both distances are large compared to $2GM$, we can expand the formula in a series up to first order:

$$\frac{\lambda_R}{\lambda_E} \approx \left(1-\frac{GM}{r_R}\right)\left(1+\frac{GM}{r_E}\right) \tag{0.4}$$

$$= 1+GM\left(\frac{1}{r_E}-\frac{1}{r_R}\right)+\ldots \tag{0.5}$$

As a further approximation, if the distance $h = r_R - r_E \ll r_E$, that is, the distance between emission and reception is small compared with the radial coordinate, then we can write

$$\frac{\lambda_R}{\lambda_E} \approx 1+GM\left(\frac{1}{r_E}-\frac{1}{r_R}\right) \tag{0.6}$$

$$= 1+GM\left(\frac{h}{r_E r_R}\right) \tag{0.7}$$

$$\approx 1+\frac{GM}{r^2}h \tag{0.8}$$

where r in the last line can be taken as the average of r_R and r_E. In this limit, we'd expect Newton's law of gravitation to apply, and a particle a distance r from a mass M experiences an acceleration $g = GM/r^2$, so we have

$$\frac{\lambda_R}{\lambda_E} \approx 1+gh \tag{0.9}$$

As an example, suppose we have a neutron star with mass $M = 3\times 10^{30}$ kg and Schwarzschild radial coordinate at the surface of $r_E = 1.2\times 10^4$ m. The redshift observed by a satellite orbiting the star at a radius $r_R = 1.7\times 10^4$ m can be calculated using the approximation formula. We need to express G in relativistic units (that is, where $c = 1$ so that GM has the units of length). Since the units of G are $\mathrm{m^3 kg^{-1} s^{-2}}$, we need to eliminate the reference to seconds which we can do by dividing by $c^2 = 9\times 10^{16}\mathrm{m^2 s^{-2}}$. That is

$$G = \frac{6.67\times 10^{-11}}{9\times 10^{16}} \tag{0.10}$$

$$= 7.41\times 10^{-28}\,\mathrm{m\ kg^{-1}} \tag{0.11}$$

For the neutron star,

$$GM = \left(7.41\times 10^{-28}\right)\left(3\times 10^{30}\right) \tag{0.12}$$

$$= 2.223\times 10^{3}\ \mathrm{m} \tag{0.13}$$

We take

$$r = \frac{1}{2}(r_E + r_R) \tag{0.14}$$
$$= 1.45 \times 10^4 \text{ m} \tag{0.15}$$
$$g = \frac{GM}{r^2} \tag{0.16}$$
$$= \frac{2.223 \times 10^3}{(1.45 \times 10^4)^2} \tag{0.17}$$
$$= 1.06 \times 10^{-5} \text{ m}^{-1} \tag{0.18}$$

Incidentally, this is a massive acceleration compared to that on the Earth's surface. In SI units, this comes out to $\left(1.06 \times 10^{-5}\right)\left(9 \times 10^{16}\right) = 9.54 \times 10^{11} \text{m s}^{-2}$.

The fractional redshift is

$$\frac{\lambda_R - \lambda_E}{\lambda_E} \approx g(r_R - r_E) \tag{0.19}$$
$$= \left(1.06 \times 10^{-5}\right)\left(5 \times 10^3\right) \tag{0.20}$$
$$= 0.0529 \tag{0.21}$$

The exact value is

$$\frac{\lambda_R - \lambda_E}{\lambda_E} = \sqrt{\frac{1 - 2GM/r_R}{1 - 2GM/r_E}} - 1 \tag{0.22}$$
$$= 0.0831 \tag{0.23}$$

This is the gravitational redshift formula. For $r_R \to \infty$, the formula reduces to

$$\frac{\lambda_R}{\lambda_E} = \frac{1}{\sqrt{1 - 2GM/r_E}} \tag{0.24}$$

This formula could be used, for example, to calculate the redshift due to a star when observed from Earth.

If both distances are large compared to $2GM$, we can expand the formula in a series up to first order:

$$\frac{\lambda_R}{\lambda_E} \approx \left(1-\frac{GM}{r_R}\right)\left(1+\frac{GM}{r_E}\right) \tag{0.25}$$

$$= 1+GM\left(\frac{1}{r_E}-\frac{1}{r_R}\right)+\ldots \tag{0.26}$$

As a further approximation, if the distance $h = r_R - r_E \ll r_E$, that is, the distance between emission and reception is small compared the radial coordinate, then we can write

$$\frac{\lambda_R}{\lambda_E} \approx 1+GM\left(\frac{1}{r_E}-\frac{1}{r_R}\right) \tag{0.27}$$

$$= 1+GM\left(\frac{h}{r_E r_R}\right) \tag{0.28}$$

$$\approx 1+\frac{GM}{r^2}h \tag{0.29}$$

where r in the last line can be taken as the average of r_R and r_E. In this limit, we'd expect Newton's law of gravitation to apply, and a particle a distance r from a mass M experiences an acceleration $g = GM/r^2$, so we have

$$\frac{\lambda_R}{\lambda_E} \approx 1+gh \tag{0.30}$$

As an example, suppose we have a neutron star with mass $M = 3\times 10^{30}$ kg and Schwarzschild radial coordinate at the surface of $r_E = 1.2\times 10^4$ m. The redshift observed by a satellite orbiting the star at a radius $r_R = 1.7\times 10^4$ m can be calculated using the approximation formula. We need to express G in relativistic units (that is, where $c = 1$ so that GM has the units of length). Since the units of G are $\mathrm{m^3 kg^{-1} s^{-2}}$, we need to eliminate the reference to seconds which we can do by dividing by $c^2 = 9\times 10^{16}\mathrm{m^2 s^{-2}}$. That is

$$G = \frac{6.67\times 10^{-11}}{9\times 10^{16}} \tag{0.31}$$

$$= 7.41\times 10^{-28}\mathrm{m\ kg^{-1}} \tag{0.32}$$

For the neutron star,

$$GM = \left(7.41\times 10^{-28}\right)\left(3\times 10^{30}\right) \tag{0.33}$$

$$= 2.223\times 10^{3}\ \mathrm{m} \tag{0.34}$$

We take

$$r = \frac{1}{2}(r_E + r_R) \tag{0.35}$$

$$= 1.45 \times 10^4 \text{ m} \tag{0.36}$$

$$g = \frac{GM}{r^2} \tag{0.37}$$

$$= \frac{2.223 \times 10^3}{(1.45 \times 10^4)^2} \tag{0.38}$$

$$= 1.06 \times 10^{-5} \text{ m}^{-1} \tag{0.39}$$

Incidentally, this is a massive acceleration compared to that on the Earth's surface. In SI units, this comes out to $\left(1.06 \times 10^{-5}\right)\left(9 \times 10^{16}\right) = 9.54 \times 10^{11} \text{m s}^{-2}$.

The fractional redshift is

$$\frac{\lambda_R - \lambda_E}{\lambda_E} \approx g(r_R - r_E) \tag{0.40}$$

$$= \left(1.06 \times 10^{-5}\right)\left(5 \times 10^3\right) \tag{0.41}$$

$$= 0.0529 \tag{0.42}$$

The exact value is

$$\frac{\lambda_R - \lambda_E}{\lambda_E} = \sqrt{\frac{1 - 2GM/r_R}{1 - 2GM/r_E}} - 1 \tag{0.43}$$

$$= 0.0831 \tag{0.44}$$

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