

SCHWARZSCHILD METRIC: GRAVITATIONAL REDSHIFT

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapters 9; Problem 9.1.

The relation between the proper time interval and Schwarzschild time interval for an object at rest is

$$(1) \quad \Delta\tau = \left(1 - \frac{2GM}{r}\right)^{1/2} \Delta t$$

We can use this relation to derive a formula for the gravitational redshift. The key to this is that the wavelength of light is $\lambda = c\Delta\tau$, where $c = 1$ is the speed of light and $\Delta\tau$ is the time interval as measured by an observer required for a single wavelength to be emitted or received. Why $\Delta\tau$ instead of Δt ? As far as I understand it, this is because τ is the only correct measure of time for an object at rest. The Schwarzschild t coordinate, although it's called the 'time coordinate', isn't really a measure of time directly.

The redshift arises because if we emit a light beam at $r = r_E$ in a direction radially outwards from the mass M and receive the light beam at $r = r_R > r_E$, then the interval Δt required for the passage of a single wavelength must be the same at both the emitter and the receiver. Why? Because the metric doesn't depend on t .

The proper time interval, however, is not the same at the two points because of the relation above. Plugging in the values, we get

$$(2) \quad \frac{\lambda_R}{\lambda_E} = \frac{\Delta\tau_R}{\Delta\tau_E}$$

$$(3) \quad = \sqrt{\frac{1 - 2GM/r_R}{1 - 2GM/r_E}}$$

This formula could be used, for example, to calculate the redshift due to a star when observed from Earth.

If both distances are large compared to $2GM$, we can expand the formula in a series up to first order:

$$(4) \quad \frac{\lambda_R}{\lambda_E} \approx \left(1 - \frac{GM}{r_R}\right) \left(1 + \frac{GM}{r_E}\right)$$

$$(5) \quad = 1 + GM \left(\frac{1}{r_E} - \frac{1}{r_R}\right) + \dots$$

As a further approximation, if the distance $h = r_R - r_E \ll r_E$, that is, the distance between emission and reception is small compared with the radial coordinate, then we can write

$$(6) \quad \frac{\lambda_R}{\lambda_E} \approx 1 + GM \left(\frac{1}{r_E} - \frac{1}{r_R}\right)$$

$$(7) \quad = 1 + GM \left(\frac{h}{r_E r_R}\right)$$

$$(8) \quad \approx 1 + \frac{GM}{r^2} h$$

where r in the last line can be taken as the average of r_R and r_E . In this limit, we'd expect Newton's law of gravitation to apply, and a particle a distance r from a mass M experiences an acceleration $g = GM/r^2$, so we have

$$(9) \quad \frac{\lambda_R}{\lambda_E} \approx 1 + gh$$

As an example, suppose we have a neutron star with mass $M = 3 \times 10^{30}$ kg and Schwarzschild radial coordinate at the surface of $r_E = 1.2 \times 10^4$ m. The redshift observed by a satellite orbiting the star at a radius $r_R = 1.7 \times 10^4$ m can be calculated using the approximation formula. We need to express G in relativistic units (that is, where $c = 1$ so that GM has the units of length). Since the units of G are $\text{m}^3 \text{kg}^{-1} \text{s}^{-2}$, we need to eliminate the reference to seconds which we can do by dividing by $c^2 = 9 \times 10^{16} \text{m}^2 \text{s}^{-2}$. That is

$$(10) \quad G = \frac{6.67 \times 10^{-11}}{9 \times 10^{16}}$$

$$(11) \quad = 7.41 \times 10^{-28} \text{m kg}^{-1}$$

For the neutron star,

$$(12) \quad GM = (7.41 \times 10^{-28}) (3 \times 10^{30})$$

$$(13) \quad = 2.223 \times 10^3 \text{m}$$

We take

$$(14) \quad r = \frac{1}{2}(r_E + r_R)$$

$$(15) \quad = 1.45 \times 10^4 \text{ m}$$

$$(16) \quad g = \frac{GM}{r^2}$$

$$(17) \quad = \frac{2.223 \times 10^3}{(1.45 \times 10^4)^2}$$

$$(18) \quad = 1.06 \times 10^{-5} \text{ m}^{-1}$$

Incidentally, this is a massive acceleration compared to that on the Earth's surface. In SI units, this comes out to $(1.06 \times 10^{-5})(9 \times 10^{16}) = 9.54 \times 10^{11} \text{ m s}^{-2}$.

The fractional redshift is

$$(19) \quad \frac{\lambda_R - \lambda_E}{\lambda_E} \approx g(r_R - r_E)$$

$$(20) \quad = (1.06 \times 10^{-5})(5 \times 10^3)$$

$$(21) \quad = 0.0529$$

The exact value is

$$(22) \quad \frac{\lambda_R - \lambda_E}{\lambda_E} = \sqrt{\frac{1 - 2GM/r_R}{1 - 2GM/r_E}} - 1$$

$$(23) \quad = 0.0831$$

This is the gravitational redshift formula. For $r_R \rightarrow \infty$, the formula reduces to

$$(24) \quad \frac{\lambda_R}{\lambda_E} = \frac{1}{\sqrt{1 - 2GM/r_E}}$$

This formula could be used, for example, to calculate the redshift due to a star when observed from Earth.

If both distances are large compared to $2GM$, we can expand the formula in a series up to first order:

$$(25) \quad \frac{\lambda_R}{\lambda_E} \approx \left(1 - \frac{GM}{r_R}\right) \left(1 + \frac{GM}{r_E}\right)$$

$$(26) \quad = 1 + GM \left(\frac{1}{r_E} - \frac{1}{r_R} \right) + \dots$$

As a further approximation, if the distance $h = r_R - r_E \ll r_E$, that is, the distance between emission and reception is small compared the radial coordinate, then we can write

$$(27) \quad \frac{\lambda_R}{\lambda_E} \approx 1 + GM \left(\frac{1}{r_E} - \frac{1}{r_R} \right)$$

$$(28) \quad = 1 + GM \left(\frac{h}{r_E r_R} \right)$$

$$(29) \quad \approx 1 + \frac{GM}{r^2} h$$

where r in the last line can be taken as the average of r_R and r_E . In this limit, we'd expect Newton's law of gravitation to apply, and a particle a distance r from a mass M experiences an acceleration $g = GM/r^2$, so we have

$$(30) \quad \frac{\lambda_R}{\lambda_E} \approx 1 + gh$$

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For the neutron star,

$$(33) \quad GM = (7.41 \times 10^{-28}) (3 \times 10^{30})$$

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We take

$$(35) \quad r = \frac{1}{2}(r_E + r_R)$$

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The fractional redshift is

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$$(41) \quad = (1.06 \times 10^{-5})(5 \times 10^3)$$

$$(42) \quad = 0.0529$$

The exact value is

$$(43) \quad \frac{\lambda_R - \lambda_E}{\lambda_E} = \sqrt{\frac{1 - 2GM/r_R}{1 - 2GM/r_E}} - 1$$

$$(44) \quad = 0.0831$$

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