SCHWARZSCHILD METRIC: RADIAL COORDINATE IS CIRCUMFERENTIAL

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 9; Box 9.1; Problem 9.4.

It's time to have a look at a proper general relativistic metric and apply some of the theory we've done up to now. To do this properly, we really should derive the Einstein equation for gravitation, but that would require a lot more discussion of tensor calculus. We can actually specify a *solution* to the Einstein equation in the form of a metric for curved space-time and leave the proof that this metric is a solution until later, when we've done a lot more theory.

The metric we'll look at was discovered by Karl Schwarzschild in 1916, very soon after Einstein developed general relativity. It describes space-time in a vacuum outside a spherically symmetric mass. The metric is

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (1)$$

The parameter r_s is known as the *Schwarzschild radius* and will turn out to depend on the mass of the object.

This metric bears a superficial resemblance to the ordinary spherical metric in flat space, which is

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (2)$$

We need to be careful in drawing any parallels between the two metrics, however, since the coordinates r and t don't have the same meaning in the two cases. To illustrate this, we'll have a look at the radial coordinate.

Suppose we look at the set of points (events) where t, r and θ are all constant, and take $\theta = \pi/2$. This specifies the equator on a sphere. In that case

$$ds = r d\phi \quad (3)$$

and if we integrate this around the complete circle (for ϕ from 0 to 2π), we get, not surprisingly $s = 2\pi r$. If we define this distance as the circumference C then

$$r = \frac{C}{2\pi} \tag{4}$$

This is hardly an earth-shattering revelation, but now for the sting in the tail. In flat 3-d space, the distance r is the radius of the circle with circumference C. In the Schwarzschild metric, this is *not* the case.

To see this, suppose we now consider a path where $dt = d\theta = d\phi = 0$ and r varies between r_A and r_B. From the metric, the path length is

$$ds = \frac{dr}{\sqrt{1 - r_s/r}} \tag{5}$$

Integrating this using software we get

$$s = \sqrt{r(r - r_s)} + \frac{r_s}{2} \ln\left[\sqrt{r(r - r_s)} + r - \frac{r_s}{2}\right] + \alpha \tag{6}$$

where α is a constant of integration.

The logarithm can be converted to an inverse hyperbolic tangent. We start with the identity

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} \tag{7}$$

If we try $x = \sqrt{1 - \frac{r_s}{r}}$ we get

$$\begin{aligned}
\tanh^{-1}\sqrt{1 - \frac{r_s}{r}} &= \frac{1}{2}\ln\frac{1+\sqrt{1-\frac{r_s}{r}}}{1-\sqrt{1-\frac{r_s}{r}}} && (8)\\
&= \frac{1}{2}\ln\frac{\left(1+\sqrt{1-\frac{r_s}{r}}\right)^2}{r_s/r} && (9)\\
&= \frac{1}{2}\ln\left[\frac{r}{r_s}\left(2 - \frac{r_s}{r} + 2\sqrt{1-\frac{r_s}{r}}\right)\right] && (10)\\
&= \frac{1}{2}\ln\left[2\frac{r}{r_s} + 2\frac{r}{r_s}\sqrt{1-\frac{r_s}{r}} - 1\right] && (11)\\
&= \frac{1}{2}\ln\left[\frac{2}{r_s}\left(\sqrt{r(r-r_s)} + r - \frac{r_s}{2}\right)\right] && (12)\\
&= \frac{1}{2}\ln\left(\sqrt{r(r-r_s)} + r - \frac{r_s}{2}\right) + \frac{1}{2}\ln\frac{2}{r_s} && (13)
\end{aligned}$$

When we evaluate this between two limits, the last term cancels out, so the path length between r_A and r_B is

$$\Delta s = \left[\sqrt{r(r-r_s)} + r_s \tanh^{-1}\sqrt{1-\frac{r_s}{r}}\right]_{r_A}^{r_B} \tag{14}$$

$$= \left[\sqrt{r(r-r_s)} + \frac{r_s}{2}\ln\left[\sqrt{r(r-r_s)} + r - \frac{r_s}{2}\right]\right]_{r_A}^{r_B} \tag{15}$$

For $r_s = 0$, this reduces to $\Delta s = r_B - r_A$, which is what we'd expect for flat space. However, if $r_s \neq 0$, this expression is clearly not just the difference in radial coordinates.

For $r_s \ll r$, we can expand this result in a series (again, using software). We get

$$\Delta s = (r_B - r_A) + \frac{r_s}{2}\ln\frac{r_B}{r_A} + \frac{3r_s^2}{8}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) + \mathcal{O}\left(r_s^3\right) \tag{16}$$

Assuming $r_B > r_A$, these first 3 terms are all positive (in fact, all the terms up to $\mathcal{O}\left(r_s^5\right)$ are positive), so the path length $\Delta s > r_B - r_A$. This means that if we take two equatorial circles with one at $r = r_A$ and the second at $r = r_B$, then the circumferences of these two circles are $2\pi r_A$ and $2\pi r_B$, but the radial distance between the circles is *greater than* $r_B - r_A$. This is one of the curious effects of using curved space-time.

For this reason, the Schwarzschild radial coordinate is a *circumferential radial coordinate*, that is, if we integrate around a full circle at radius r, we get a circumference of $2\pi r$, but if we measure the distance outwards from $r = r_A$ to $r = r_B$, we don't get $r_B - r_A$.

If we apply this to the Earth, we can find how much the Schwarzschild radial coordinate differs from the Euclidean radius. If we start at the Earth's surface where $r_A = 6.38 \times 10^6$ m and move radially outward by 100 km to $r_B = 6.48 \times 10^6$ m, then using $\frac{r_s}{2} = GM = 4.44 \times 10^{-3}$ m for the Earth, the first order in r_s deviation term above is

$$GM \ln\frac{r_B}{r_A} = 6.9 \times 10^{-5} \text{ m} \tag{17}$$

which is under a tenth of a millimetre. Although small, this is far from an unmeasurable deviation.

PINGBACKS

Pingback: Spherical metric: distance in 2-d curved space
Pingback: Schwarzschild radius
Pingback: Schwarzschild metric: time coordinate
Pingback: Particle orbits - conserved quantities
Pingback: Photon equations of motion
Pingback: Event horizon: distance from external radius up to r = 2GM

Pingback: Event horizon: proper time to fall into r = 0

Pingback: Event horizon: time and space swap round

Pingback: Painlevé-Gullstrand (global rain) coordinates

Pingback: Painlevé-Gullstrand (global rain) coordinates

Pingback: Christoffel symbols for Schwarzschild metric

Pingback: Einstein equation solution for the interior of a spherically symmetric star

Pingback: Cosmic strings