SCHWARZSCHILD METRIC: FOUR-MOMENTUM OF A PHOTON

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapters 9; Problem 9.6.

This is a first example of the use of the time component of the Schwarzschild metric. This metric is, for a spherical mass M:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{1}$$

Suppose we have an observer at a Schwarzschild radius R from the centre of a star of mass M, and this observer watches a photon move radially outward. The observer measures the energy of the photon to be E. We can use this to calculate the four-momentum of the photon.

In special relativity, for an observer at rest the observer's four-velocity is $u^i = [1,0,0,0]$ so the scalar product of the observer's four-velocity with another object's momentum (as measured by the observer) is

$$\begin{aligned}
\mathbf{p}\cdot\mathbf{u}_{obs} &= g_{ij}p^i u^j & (2)\\
&= -p^t u^t & (3)\\
&= -p^t & (4)\\
&= -E & (5)
\end{aligned}$$

since the time component of an object's four-momentum is its energy. Since this is a tensor equation, it should be true in curved space-time as well. In the Schwarzschild metric, an observer at rest has

$$u^t = \left[\left(1-\frac{2GM}{R}\right)^{-1/2},0,0,0\right] \tag{6}$$

Therefore, we get

$$\begin{aligned}
\mathbf{p}\cdot\mathbf{u}_{obs} &= g_{ij}p^{i}u^{j} &(7)\\
&= -\left(1-\frac{2GM}{R}\right)p^{t}\left(1-\frac{2GM}{R}\right)^{-1/2} &(8)\\
&= -p^{t}\left(1-\frac{2GM}{R}\right)^{1/2} &(9)\\
&= -E &(10)\\
p^{t} &= E\left(1-\frac{2GM}{R}\right)^{-1/2} &(11)
\end{aligned}$$

For a photon, $\mathbf{p}\cdot\mathbf{p}=0$, and for a photon moving in the radial direction $p^{\theta}=p^{\phi}=0$ so

$$\mathbf{p}\cdot\mathbf{p}=g_{ij}p^{i}p^{j} \tag{12}$$

$$0=-\left(1-\frac{2GM}{R}\right)E^{2}\left(1-\frac{2GM}{R}\right)^{-1}+\left(1-\frac{2GM}{R}\right)^{-1}(p^{r})^{2} \tag{13}$$

$$0=-E^{2}+\left(1-\frac{2GM}{R}\right)^{-1}(p^{r})^{2} \tag{14}$$

$$p^{r}=E\sqrt{1-\frac{2GM}{R}} \tag{15}$$

Thus the photon's four-momentum in the Schwarzschild basis is

$$\mathbf{p}=\left[E\left(1-\frac{2GM}{R}\right)^{-1/2},E\sqrt{1-\frac{2GM}{R}},0,0\right] \tag{16}$$

PINGBACKS

Pingback: Particles falling towards a mass
Pingback: Particle falling towards a mass: two types of velocity
Pingback: Vertical particle motion
Pingback: Orbit of a comet around a black hole
Pingback: Schwarzschild metric equivalent to weak field solution for spherical object
Pingback: Spherically symmetric solution to the Einstein equation