SCHWARZSCHILD METRIC: FOUR-MOMENTUM OF A PHOTON

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapters 9; Problem 9.6.

This is a first example of the use of the time component of the Schwarzschild metric. This metric is, for a spherical mass M:

(1)
$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

Suppose we have an observer at a Schwarzschild radius R from the centre of a star of mass M, and this observer watches a photon move radially outward. The observer measures the energy of the photon to be E. We can use this to calculate the four-momentum of the photon.

In special relativity, for an observer at rest the observer's four-velocity is $u^i = [1,0,0,0]$ so the scalar product of the observer's four-velocity with another object's momentum (as measured by the observer) is

$$\begin{aligned}
\mathbf{p}\cdot\mathbf{u}_{obs} &= g_{ij}p^i u^j && (2)\\
&= -p^t u^t && (3)\\
&= -p^t && (4)\\
&= -E && (5)
\end{aligned}$$

since the time component of an object's four-momentum is its energy. Since this is a tensor equation, it should be true in curved space-time as well. In the Schwarzschild metric, an observer at rest has

$$u^t = \left[\left(1-\frac{2GM}{R}\right)^{-1/2}, 0, 0, 0\right] \tag{6}$$

Therefore, we get

$$\mathbf{p}\cdot\mathbf{u}_{obs} = g_{ij}p^i u^j \tag{7}$$

$$= -\left(1-\frac{2GM}{R}\right)p^t\left(1-\frac{2GM}{R}\right)^{-1/2} \tag{8}$$

$$= -p^t\left(1-\frac{2GM}{R}\right)^{1/2} \tag{9}$$

$$= -E \tag{10}$$

$$p^t = E\left(1-\frac{2GM}{R}\right)^{-1/2} \tag{11}$$

For a photon, $\mathbf{p}\cdot\mathbf{p}=0$, and for a photon moving in the radial direction $p^\theta = p^\phi = 0$ so

$$\mathbf{p}\cdot\mathbf{p} = g_{ij}p^i p^j \tag{12}$$

$$0 = -\left(1-\frac{2GM}{R}\right)E^2\left(1-\frac{2GM}{R}\right)^{-1} + \left(1-\frac{2GM}{R}\right)^{-1}(p^r)^2 \tag{13}$$

$$0 = -E^2 + \left(1-\frac{2GM}{R}\right)^{-1}(p^r)^2 \tag{14}$$

$$p^r = E\sqrt{1-\frac{2GM}{R}} \tag{15}$$

Thus the photon's four-momentum in the Schwarzschild basis is

$$\mathbf{p} = \left[E\left(1-\frac{2GM}{R}\right)^{-1/2}, E\sqrt{1-\frac{2GM}{R}}, 0, 0\right] \tag{16}$$

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