

SCHWARZSCHILD METRIC: FOUR-MOMENTUM OF A PHOTON

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapters 9; Problem 9.6.

This is a first example of the use of the time component of the Schwarzschild metric. This metric is, for a spherical mass M :

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (1)$$

Suppose we have an observer at a Schwarzschild radius R from the centre of a star of mass M , and this observer watches a photon move radially outward. The observer measures the energy of the photon to be E . We can use this to calculate the four-momentum of the photon.

In special relativity, for an observer at rest the observer's four-velocity is $u^i = [1, 0, 0, 0]$ so the scalar product of the observer's four-velocity with another object's momentum (as measured by the observer) is

$$\mathbf{p} \cdot \mathbf{u}_{obs} = g_{ij} p^i u^j \quad (2)$$

$$= -p^t u^t \quad (3)$$

$$= -p^t \quad (4)$$

$$= -E \quad (5)$$

since the time component of an object's four-momentum is its energy. Since this is a tensor equation, it should be true in curved space-time as well. In the Schwarzschild metric, an observer at rest has

$$u^t = \left[\left(1 - \frac{2GM}{R}\right)^{-1/2}, 0, 0, 0 \right] \quad (6)$$

Therefore, we get

$$\mathbf{p} \cdot \mathbf{u}_{obs} = g_{ij} p^i u^j \quad (7)$$

$$= - \left(1 - \frac{2GM}{R}\right) p^t \left(1 - \frac{2GM}{R}\right)^{-1/2} \quad (8)$$

$$= -p^t \left(1 - \frac{2GM}{R}\right)^{1/2} \quad (9)$$

$$= -E \quad (10)$$

$$p^t = E \left(1 - \frac{2GM}{R}\right)^{-1/2} \quad (11)$$

For a photon, $\mathbf{p} \cdot \mathbf{p} = 0$, and for a photon moving in the radial direction $p^\theta = p^\phi = 0$ so

$$\mathbf{p} \cdot \mathbf{p} = g_{ij} p^i p^j \quad (12)$$

$$0 = - \left(1 - \frac{2GM}{R}\right) E^2 \left(1 - \frac{2GM}{R}\right)^{-1} + \left(1 - \frac{2GM}{R}\right)^{-1} (p^r)^2 \quad (13)$$

$$0 = -E^2 + \left(1 - \frac{2GM}{R}\right)^{-1} (p^r)^2 \quad (14)$$

$$p^r = E \sqrt{1 - \frac{2GM}{R}} \quad (15)$$

Thus the photon's four-momentum in the Schwarzschild basis is

$$\mathbf{p} = \left[E \left(1 - \frac{2GM}{R}\right)^{-1/2}, E \sqrt{1 - \frac{2GM}{R}}, 0, 0 \right] \quad (16)$$

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