PARTICLES FALLING TOWARDS A MASS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problem 10.3.

Another example of using the Schwarzschild metric to calculate some properties of a particle's trajectory. This time we start off with two masses at infinity, with one at rest (so $e=1$) and another moving radially inwards towards the central mass M with $e=2$. What are the speeds of these two particles when they pass the point $r=6GM$?

We can start by looking at the invariant equation

$$E=-\mathbf{p}\cdot\mathbf{u}_{\text{obs}} \tag{1}$$

where $\mathbf{p}$ is the particle's momentum as measured by an observer at rest at $r=6GM$, and $\mathbf{u}_{\text{obs}}$ is the four-velocity of the observer. In the Schwarzschild metric an observer at rest has four-velocity

$$u^t=\left[\left(1-\frac{2GM}{R}\right)^{-1/2},0,0,0\right] \tag{2}$$

The relation above then becomes

$$\begin{aligned}
E &= -g_{tt}p^t u^t && (3)\\
&= \left(1-\frac{2GM}{r}\right)\left(m\frac{dt}{d\tau}\right)\left(1-\frac{2GM}{r}\right)^{-1/2} && (4)\\
&= m\left(1-\frac{2GM}{r}\right)^{1/2}\frac{dt}{d\tau} && (5)
\end{aligned}$$

where m is the particle's mass.

We know that the conserved quantity e is given by

$$\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau}=e \tag{6}$$

so we can substitute for $dt/d\tau$ above to get

$$E = m\left(1-\frac{2GM}{r}\right)^{1/2}\frac{e}{\left(1-\frac{2GM}{r}\right)} \tag{7}$$

$$= me\left(1-\frac{2GM}{r}\right)^{-1/2} \tag{8}$$

The energy of a particle moving at speed v is (from special relativity)

$$E = \frac{m}{\sqrt{1-v^2}} \tag{9}$$

so combining the two results we get

$$\frac{m}{\sqrt{1-v^2}} = me\left(1-\frac{2GM}{r}\right)^{-1/2} \tag{10}$$

$$\frac{1}{e^2}\left(1-\frac{2GM}{r}\right) = 1-v^2 \tag{11}$$

$$v = \left[1-\frac{1}{e^2}\left(1-\frac{2GM}{r}\right)\right]^{1/2} \tag{12}$$

For $e=1$ and $r=6GM$:

$$v = \frac{1}{\sqrt{3}} \tag{13}$$

For $e=2$:

$$v = \sqrt{\frac{5}{6}} \tag{14}$$

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