

PARTICLES FALLING TOWARDS A MASS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problem 10.3.

Another example of using the Schwarzschild metric to calculate some properties of a particle's trajectory. This time we start off with two masses at infinity, with one at rest (so $e = 1$) and another moving radially inwards towards the central mass M with $e = 2$. What are the speeds of these two particles when they pass the point $r = 6GM$?

We can start by looking at the invariant equation

$$(0.1) \quad E = -\mathbf{p} \cdot \mathbf{u}_{\text{obs}}$$

where \mathbf{p} is the particle's momentum as measured by an observer at rest at $r = 6GM$, and \mathbf{u}_{obs} is the four-velocity of the observer. In the Schwarzschild metric an observer at rest has four-velocity

$$(0.2) \quad u^t = \left[\left(1 - \frac{2GM}{R}\right)^{-1/2}, 0, 0, 0 \right]$$

The relation above then becomes

$$(0.3) \quad E = -g_{tt} p^t u^t$$

$$(0.4) \quad = \left(1 - \frac{2GM}{r}\right) \left(m \frac{dt}{d\tau}\right) \left(1 - \frac{2GM}{r}\right)^{-1/2}$$

$$(0.5) \quad = m \left(1 - \frac{2GM}{r}\right)^{1/2} \frac{dt}{d\tau}$$

where m is the particle's mass.

We know that the conserved quantity e is given by

$$(0.6) \quad \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau} = e$$

so we can substitute for $dt/d\tau$ above to get

$$(0.7) \quad E = m \left(1 - \frac{2GM}{r}\right)^{1/2} \frac{e}{\left(1 - \frac{2GM}{r}\right)}$$

$$(0.8) \quad = me \left(1 - \frac{2GM}{r}\right)^{-1/2}$$

The energy of a particle moving at speed v is (from special relativity)

$$(0.9) \quad E = \frac{m}{\sqrt{1-v^2}}$$

so combining the two results we get

$$(0.10) \quad \frac{m}{\sqrt{1-v^2}} = me \left(1 - \frac{2GM}{r}\right)^{-1/2}$$

$$(0.11) \quad \frac{1}{e^2} \left(1 - \frac{2GM}{r}\right) = 1 - v^2$$

$$(0.12) \quad v = \left[1 - \frac{1}{e^2} \left(1 - \frac{2GM}{r}\right)\right]^{1/2}$$

For $e = 1$ and $r = 6GM$:

$$(0.13) \quad v = \frac{1}{\sqrt{3}}$$

For $e = 2$:

$$(0.14) \quad v = \sqrt{\frac{5}{6}}$$

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