

PARTICLES FALLING TOWARDS A MASS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problem 10.3.

Another example of using the Schwarzschild metric to calculate some properties of a particle's trajectory. This time we start off with two masses at infinity, with one at rest (so $e = 1$) and another moving radially inwards towards the central mass M with $e = 2$. What are the speeds of these two particles when they pass the point $r = 6GM$?

We can start by looking at the invariant equation

$$E = -\mathbf{p} \cdot \mathbf{u}_{\text{obs}} \quad (1)$$

where \mathbf{p} is the particle's momentum as measured by an observer at rest at $r = 6GM$, and \mathbf{u}_{obs} is the four-velocity of the observer. In the Schwarzschild metric an observer at rest has four-velocity

$$u^t = \left[\left(1 - \frac{2GM}{R} \right)^{-1/2}, 0, 0, 0 \right] \quad (2)$$

The relation above then becomes

$$E = -g_{tt} p^t u^t \quad (3)$$

$$= \left(1 - \frac{2GM}{r} \right) \left(m \frac{dt}{d\tau} \right) \left(1 - \frac{2GM}{r} \right)^{-1/2} \quad (4)$$

$$= m \left(1 - \frac{2GM}{r} \right)^{1/2} \frac{dt}{d\tau} \quad (5)$$

where m is the particle's mass.

We know that the conserved quantity e is given by

$$\left(1 - \frac{2GM}{r} \right) \frac{dt}{d\tau} = e \quad (6)$$

so we can substitute for $dt/d\tau$ above to get

$$E = m \left(1 - \frac{2GM}{r}\right)^{1/2} \frac{e}{\left(1 - \frac{2GM}{r}\right)} \quad (7)$$

$$= me \left(1 - \frac{2GM}{r}\right)^{-1/2} \quad (8)$$

The energy of a particle moving at speed v is (from special relativity)

$$E = \frac{m}{\sqrt{1-v^2}} \quad (9)$$

so combining the two results we get

$$\frac{m}{\sqrt{1-v^2}} = me \left(1 - \frac{2GM}{r}\right)^{-1/2} \quad (10)$$

$$\frac{1}{e^2} \left(1 - \frac{2GM}{r}\right) = 1 - v^2 \quad (11)$$

$$v = \left[1 - \frac{1}{e^2} \left(1 - \frac{2GM}{r}\right)\right]^{1/2} \quad (12)$$

For $e = 1$ and $r = 6GM$:

$$v = \frac{1}{\sqrt{3}} \quad (13)$$

For $e = 2$:

$$v = \sqrt{\frac{5}{6}} \quad (14)$$

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