

PARTICLE FALLING TOWARDS A MASS: TWO TYPES OF VELOCITY

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problem 10.4.

The radial equation of motion in the Schwarzschild metric is

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 + \frac{1}{2} \frac{l^2}{r^2} - GM \left(\frac{1}{r} + \frac{l^2}{r^3} \right) = \frac{1}{2} (e^2 - 1) \quad (1)$$

We can use this to derive an equation for dr/dt , the rate of change of r with respect to the Schwarzschild time coordinate. The coordinate t isn't the time as measured by any particular object (that time is the proper time τ in the reference frame of the object) so we wouldn't expect it to be the same as $dr/d\tau$.

To get the equation, we can use

$$\frac{dr}{d\tau} = \frac{dr}{dt} \frac{dt}{d\tau} \quad (2)$$

$$= \frac{dr}{dt} e \left(1 - \frac{2GM}{r} \right)^{-1} \quad (3)$$

where the last line uses the definition of e . Plugging this into the top equation we get

$$\frac{dr}{dt} = \frac{1}{e} \left(1 - \frac{2GM}{r} \right) \left[e^2 - 1 + 2GM \left(\frac{1}{r} + \frac{l^2}{r^3} \right) - \frac{l^2}{r^2} \right]^{1/2} \quad (4)$$

As r approaches $2GM$, $dr/dt \rightarrow 0$.

In the special case where we drop an object from rest at $r = r_0$, we can work out both dr/dt and $dr/d\tau$. In this case, motion is radially inward so $l = 0$. To find e , we use the fact that for an object at rest at $r = r_0$:

$$e = \left(1 - \frac{2GM}{r_0}\right) \frac{dt}{d\tau} \quad (5)$$

$$= \left(1 - \frac{2GM}{r_0}\right) u^t \quad (6)$$

$$= \left(1 - \frac{2GM}{r_0}\right) \left(1 - \frac{2GM}{r_0}\right)^{-1/2} \quad (7)$$

$$= \left(1 - \frac{2GM}{r_0}\right)^{1/2} \quad (8)$$

We have therefore

$$\frac{dr}{dt} = \frac{1}{e} \left(1 - \frac{2GM}{r}\right) \left(e^2 - 1 + \frac{2GM}{r}\right)^{1/2} \quad (9)$$

$$= \frac{1 - \frac{2GM}{r}}{\sqrt{1 - \frac{2GM}{r_0}}} \sqrt{2GM \left(\frac{1}{r} - \frac{1}{r_0}\right)} \quad (10)$$

$$= \left(1 - \frac{2GM}{r}\right) \sqrt{\frac{2GM}{r}} \sqrt{\frac{r_0 - r}{r_0 - 2GM}} \quad (11)$$

From 1 we get

$$\frac{dr}{d\tau} = \sqrt{e^2 - 1 + \frac{2GM}{r}} \quad (12)$$

$$= \sqrt{2GM \left(\frac{1}{r} - \frac{1}{r_0}\right)} \quad (13)$$

Comparing the two, we see that

$$\frac{dr}{dt} = \frac{1 - \frac{2GM}{r}}{\sqrt{1 - \frac{2GM}{r_0}}} \frac{dr}{d\tau} \quad (14)$$

For the case where the object is released from rest at $r_0 = \infty$, the speed at $r = 6GM$ is

$$\frac{dr}{d\tau} = \frac{1}{\sqrt{3}} \quad (15)$$

which agrees with the earlier calculation done by a different method.