VERTICAL PARTICLE MOTION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problem 10.5.

And now for a general relativistic look at the standard physics problem of throwing an object up in a gravitational field. Suppose we start at radius $r = r_0$ and throw up the object so that it comes to rest momentarily at $r = r_1$ before turning around and falling back to r_0. What is the total proper time (as measured by the object) in this trip?

We begin by working out the energy e. For an object at rest at $r = r_1$:

$$e = \left(1-\frac{2GM}{r_1}\right)\frac{dt}{d\tau} \tag{0.1}$$

$$= \left(1-\frac{2GM}{r_1}\right)u^t \tag{0.2}$$

$$= \left(1-\frac{2GM}{r_1}\right)\left(1-\frac{2GM}{r_1}\right)^{-1/2} \tag{0.3}$$

$$= \left(1-\frac{2GM}{r_1}\right)^{1/2} \tag{0.4}$$

The radial equation of motion in the Schwarzschild metric is

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+\frac{1}{2}\frac{l^2}{r^2}-GM\left(\frac{1}{r}+\frac{l^2}{r^3}\right)=\frac{1}{2}\left(e^2-1\right) \tag{0.5}$$

For radial motion, $l = 0$ so we get

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(e^2-1\right)+\frac{GM}{r} \tag{0.6}$$

$$= GM\left(\frac{1}{r}-\frac{1}{r_1}\right) \tag{0.7}$$

$$= GM\frac{r_1-r}{rr_1} \tag{0.8}$$

To find the time that elapses on the upward leg of the journey, we must evaluate

$$\Delta\tau = \sqrt{\frac{r_1}{2GM}} \int_{r_0}^{r_1} \sqrt{\frac{r}{r_1 - r}} dr \tag{0.9}$$

This is, as Moore says, a bit of a nasty integral. Using software produces a bit of a jumble, so I resorted to the old-fashioned method of looking the integral up in a table, which produced a somewhat nicer result. We get

$$\Delta\tau = \sqrt{\frac{r_1}{2GM}} \left[-\sqrt{r(r_1 - r)} - r_1 \arctan\left(-\sqrt{\frac{r}{r_1 - r}} \right) \right]_{r_0}^{r_1} \tag{0.10}$$

$$= \sqrt{\frac{r_1}{2GM}} \left[-\sqrt{r(r_1 - r)} + r_1 \arctan\sqrt{\frac{r}{r_1 - r}} \right]_{r_0}^{r_1} \tag{0.11}$$

At the upper limit, the first term is zero and the second term is

$$r_1 \arctan(\infty) = \frac{\pi}{2} r_1 \tag{0.12}$$

so the result is

$$\Delta\tau = \sqrt{\frac{r_1}{2GM}} \left[\sqrt{r_0(r_1 - r_0)} - r_1 \arctan\sqrt{\frac{r_0}{r_1 - r_0}} + \frac{\pi}{2} r_1 \right] \tag{0.13}$$

Here we're assuming that the arctan lies in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The total time is twice this, so

$$\tau = \sqrt{\frac{2r_1}{GM}} \left[\sqrt{r_0(r_1 - r_0)} - r_1 \arctan\sqrt{\frac{r_0}{r_1 - r_0}} + \frac{\pi}{2} r_1 \right] \tag{0.14}$$