VERTICAL PARTICLE MOTION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problem 10.5.

And now for a general relativistic look at the standard physics problem of throwing an object up in a gravitational field. Suppose we start at radius $r = r_0$ and throw up the object so that it comes to rest momentarily at $r = r_1$ before turning around and falling back to r_0. What is the total proper time (as measured by the object) in this trip?

We begin by working out the energy e. For an object at rest at $r = r_1$:

$$e = \left(1 - \frac{2GM}{r_1}\right)\frac{dt}{d\tau} \tag{1}$$

$$= \left(1 - \frac{2GM}{r_1}\right)u^t \tag{2}$$

$$= \left(1 - \frac{2GM}{r_1}\right)\left(1 - \frac{2GM}{r_1}\right)^{-1/2} \tag{3}$$

$$= \left(1 - \frac{2GM}{r_1}\right)^{1/2} \tag{4}$$

The radial equation of motion in the Schwarzschild metric is

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{l^2}{r^2} - GM\left(\frac{1}{r} + \frac{l^2}{r^3}\right) = \frac{1}{2}\left(e^2 - 1\right) \tag{5}$$

For radial motion, $l = 0$ so we get

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(e^2 - 1\right) + \frac{GM}{r} \tag{6}$$

$$= GM\left(\frac{1}{r} - \frac{1}{r_1}\right) \tag{7}$$

$$= GM\frac{r_1 - r}{rr_1} \tag{8}$$

To find the time that elapses on the upward leg of the journey, we must evaluate

$$\Delta\tau=\sqrt{\frac{r_1}{2GM}}\int_{r_0}^{r_1}\sqrt{\frac{r}{r_1-r}}dr \tag{9}$$

This is, as Moore says, a bit of a nasty integral. Using software produces a bit of a jumble, so I resorted to the old-fashioned method of looking the integral up in a table, which produced a somewhat nicer result. We get

$$\Delta\tau=\sqrt{\frac{r_1}{2GM}}\left[-\sqrt{r\left(r_1-r\right)}-r_1\arctan\left(-\sqrt{\frac{r}{r_1-r}}\right)\right]_{r_0}^{r_1} \tag{10}$$

$$=\sqrt{\frac{r_1}{2GM}}\left[-\sqrt{r\left(r_1-r\right)}+r_1\arctan\sqrt{\frac{r}{r_1-r}}\right]_{r_0}^{r_1} \tag{11}$$

At the upper limit, the first term is zero and the second term is

$$r_1\arctan\left(\infty\right)=\frac{\pi}{2}r_1 \tag{12}$$

so the result is

$$\Delta\tau=\sqrt{\frac{r_1}{2GM}}\left[\sqrt{r_0\left(r_1-r_0\right)}-r_1\arctan\sqrt{\frac{r_0}{r_1-r_0}}+\frac{\pi}{2}r_1\right] \tag{13}$$

Here we're assuming that the arctan lies in the range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. The total time is twice this, so

$$\tau=\sqrt{\frac{2r_1}{GM}}\left[\sqrt{r_0\left(r_1-r_0\right)}-r_1\arctan\sqrt{\frac{r_0}{r_1-r_0}}+\frac{\pi}{2}r_1\right] \tag{14}$$