CIRCULAR ORBITS: RELATION BETWEEN RADIUS AND ANGULAR MOMENTUM

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problems 10.7-8.

For a circular orbit, the relation between the angular momentum and radius is

$$r=\frac{6GM}{1\mp\sqrt{1-12\left(GM/l\right)^{2}}} \tag{0.1}$$

We can invert this to get l in terms of r:

$$\begin{aligned}
\mp\sqrt{1-12\left(GM/l\right)^{2}} &= \frac{6GM}{r}-1 && (0.2)\\
-12\frac{G^{2}M^{2}}{l^{2}} &= \left(\frac{6GM}{r}-1\right)^{2}-1 && (0.3)\\
l^{2} &= \frac{12r^{2}\left(GM\right)^{2}}{r^{2}-\left(6GM-r\right)^{2}} && (0.4)\\
&= \frac{r^{2}GM}{r-3GM} && (0.5)
\end{aligned}$$

From this we can get an expression for $\tilde{E}$ for a circular orbit where $dr/d\tau=0$:

$$\tilde{E} = \frac{1}{2}\frac{l^2}{r^2} - GM\left(\frac{1}{r} + \frac{l^2}{r^3}\right) \tag{0.6}$$

$$= -\frac{GM}{r} + \frac{r^2 GM}{r-3GM}\left(\frac{1}{2r^2} - \frac{GM}{r^3}\right) \tag{0.7}$$

$$= -\frac{GM}{2r}\left[2 - \frac{2r^3}{r-3GM}\left(\frac{1}{2r^2} - \frac{GM}{r^3}\right)\right] \tag{0.8}$$

$$= -\frac{GM}{2r}\left[\frac{r}{r-3GM}\left(\frac{2r-6GM}{r} - \left(1-\frac{2GM}{r}\right)\right)\right] \tag{0.9}$$

$$= -\frac{GM}{2r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right) \tag{0.10}$$

The energy per unit mass e is then, using $\tilde{E} = \frac{1}{2}\left(e^2-1\right)$:

$$e = \left[1 - \frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{1/2} \tag{0.11}$$

For $r = 6GM$, this comes out to $e = \sqrt{\frac{8}{9}}$.

As an example of the use of these formulas, suppose we start an object at infinity with no radial velocity, but with an infinitesimal tangential velocity which gives it an angular momentum l. Since the object is at an infinite distance, the formula $l = r^2\omega$ means that in the limit as $r \to \infty$, $\omega \to 0$ so that the product $r^2\omega = l$, thus the tangential motion really *is* infinitesimal.

As the object falls in towards the central mass, it spirals in, keeping l constant. Since the object started off essentially at rest, $e = 1$ so one solution is for the object to end up in a circular orbit with $r = 4GM$. From the above formula, this corresponds to

$$l = 4GM\sqrt{\frac{GM}{4GM-3GM}} \tag{0.12}$$

$$= 4GM \tag{0.13}$$

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