

CIRCULAR ORBITS: RELATION BETWEEN RADIUS AND ANGULAR MOMENTUM

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problems 10.7-8.

For a circular orbit, the relation between the angular momentum and radius is

$$r = \frac{6GM}{1 \mp \sqrt{1 - 12(GM/l)^2}} \quad (1)$$

We can invert this to get l in terms of r :

$$\mp \sqrt{1 - 12(GM/l)^2} = \frac{6GM}{r} - 1 \quad (2)$$

$$-12 \frac{G^2 M^2}{l^2} = \left(\frac{6GM}{r} - 1 \right)^2 - 1 \quad (3)$$

$$l^2 = \frac{12r^2(GM)^2}{r^2 - (6GM - r)^2} \quad (4)$$

$$= \frac{r^2 GM}{r - 3GM} \quad (5)$$

From this we can get an expression for \tilde{E} for a circular orbit where $dr/d\tau = 0$:

$$\tilde{E} = \frac{1}{2} \frac{l^2}{r^2} - GM \left(\frac{1}{r} + \frac{l^2}{r^3} \right) \quad (6)$$

$$= -\frac{GM}{r} + \frac{r^2 GM}{r-3GM} \left(\frac{1}{2r^2} - \frac{GM}{r^3} \right) \quad (7)$$

$$= -\frac{GM}{2r} \left[2 - \frac{2r^3}{r-3GM} \left(\frac{1}{2r^2} - \frac{GM}{r^3} \right) \right] \quad (8)$$

$$= -\frac{GM}{2r} \left[\frac{r}{r-3GM} \left(\frac{2r-6GM}{r} - \left(1 - \frac{2GM}{r} \right) \right) \right] \quad (9)$$

$$= -\frac{GM}{2r} \left(1 - \frac{3GM}{r} \right)^{-1} \left(1 - \frac{4GM}{r} \right) \quad (10)$$

The energy per unit mass e is then, using $\tilde{E} = \frac{1}{2}(e^2 - 1)$:

$$e = \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r} \right)^{-1} \left(1 - \frac{4GM}{r} \right) \right]^{1/2} \quad (11)$$

For $r = 6GM$, this comes out to $e = \sqrt{\frac{8}{9}}$.

As an example of the use of these formulas, suppose we start an object at infinity with no radial velocity, but with an infinitesimal tangential velocity which gives it an angular momentum l . Since the object is at an infinite distance, the formula $l = r^2 \omega$ means that in the limit as $r \rightarrow \infty$, $\omega \rightarrow 0$ so that the product $r^2 \omega = l$, thus the tangential motion really *is* infinitesimal.

As the object falls in towards the central mass, it spirals in, keeping l constant. Since the object started off essentially at rest, $e = 1$ so one solution is for the object to end up in a circular orbit with $r = 4GM$. From the above formula, this corresponds to

$$l = 4GM \sqrt{\frac{GM}{4GM - 3GM}} \quad (12)$$

$$= 4GM \quad (13)$$

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