

CIRCULAR ORBIT: SCHWARZSCHILD VS NEWTON

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Box 10.3; Problems 10.10, 10.11.

For a circular orbit, the Schwarzschild angular speed is obtained from the angular momentum:

$$(0.1) \quad \omega = \frac{d\phi}{d\tau} = \frac{l}{r^2}$$
$$(0.2) \quad = \frac{1}{r^2} \sqrt{\frac{r^2 GM}{r - 3GM}}$$

We can write this as

$$(0.3) \quad \omega^2 = \frac{GM}{r^2(r - 3GM)}$$

Comparing this with the angular speed measured at infinity $\Omega = \frac{d\phi}{dt}$ we have

$$(0.4) \quad \Omega^2 = \frac{GM}{r^3}$$

Thus $\omega > \Omega$.

The relation between r and l for a circular orbit is

$$(0.5) \quad r = \frac{6GM}{1 \mp \sqrt{1 - 12(GM/l)^2}}$$

In the Newtonian case, we can use Kepler's law at all distances, so we have

$$(0.6) \quad \Omega^2 = \frac{l^2}{r^4}$$

$$(0.7) \quad = \frac{GM}{r^3}$$

$$(0.8) \quad r = \frac{l^2}{GM}$$

Note that we can get the same result by defining the Newtonian energy

$$(0.9) \quad E_N = \frac{1}{2} \left(\frac{dr}{dt} \right)^2 + V_N$$

$$(0.10) \quad V_N = \frac{1}{2} \frac{l^2}{r^2} - \frac{GM}{r}$$

and then solving

$$(0.11) \quad \frac{dV_N}{dr} = -\frac{l^2}{r^3} + \frac{GM}{r^2} = 0$$

In the Newtonian case, the potential energy V_N has only one minimum. For large l , we can expand 0.5 to get (using the minus sign)

$$(0.12) \quad r \rightarrow \frac{6GM}{1 - \left(1 - \frac{1}{2} \frac{12(GM)^2}{l^2} \right)}$$

$$(0.13) \quad = \frac{l^2}{GM}$$

Thus the Schwarzschild case reduces to the Newtonian case for large l .