

## CIRCULAR ORBIT: SCHWARZSCHILD VS NEWTON

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Box 10.3; Problems 10.10, 10.11.

For a circular orbit, the Schwarzschild angular speed is obtained from the angular momentum:

$$\omega = \frac{d\phi}{d\tau} = \frac{l}{r^2} \quad (1)$$

$$= \frac{1}{r^2} \sqrt{\frac{r^2 GM}{r - 3GM}} \quad (2)$$

We can write this as

$$\omega^2 = \frac{GM}{r^2 (r - 3GM)} \quad (3)$$

Comparing this with the angular speed measured at infinity  $\Omega = \frac{d\phi}{dt}$  we have

$$\Omega^2 = \frac{GM}{r^3} \quad (4)$$

Thus  $\omega > \Omega$ .

The relation between  $r$  and  $l$  for a circular orbit is

$$r = \frac{6GM}{1 \mp \sqrt{1 - 12(GM/l)^2}} \quad (5)$$

In the Newtonian case, we can use Kepler's law at all distances, so we have

$$\Omega^2 = \frac{l^2}{r^4} \quad (6)$$

$$= \frac{GM}{r^3} \quad (7)$$

$$r = \frac{l^2}{GM} \quad (8)$$

Note that we can get the same result by defining the Newtonian energy

$$E_N = \frac{1}{2} \left( \frac{dr}{dt} \right)^2 + V_N \quad (9)$$

$$V_N = \frac{1}{2} \frac{l^2}{r^2} - \frac{GM}{r} \quad (10)$$

and then solving

$$\frac{dV_N}{dr} = -\frac{l^2}{r^3} + \frac{GM}{r^2} = 0 \quad (11)$$

In the Newtonian case, the potential energy  $V_N$  has only one minimum. For large  $l$ , we can expand 5 to get (using the minus sign)

$$r \rightarrow \frac{6GM}{1 - \left( 1 - \frac{1}{2} \frac{12(GM)^2}{l^2} \right)} \quad (12)$$

$$= \frac{l^2}{GM} \quad (13)$$

Thus the Schwarzschild case reduces to the Newtonian case for large  $l$ .