CIRCULAR ORBITS: 3 MEASUREMENTS OF THE PERIOD

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problem 10.12.

We can generalize an earlier problem by working out the period of a circular orbit as measured by three different observers. The first observer is attached to the orbiting object which circles the mass at a radius r. The second observer is at rest at radius r, while the third observer is at infinity.

From the formula for angular momentum as measured by the orbiting object:

$$l^2 = \frac{r^2 GM}{r-3GM} \tag{1}$$

we can calculate the angular speed from $l = r^2\omega$, so

$$\begin{aligned} \omega &= \frac{l}{r^2} & (2)\\ &= \frac{1}{r}\sqrt{\frac{GM}{r-3GM}} & (3) \end{aligned}$$

The period is then

$$\begin{aligned} T &= \frac{2\pi}{\omega} & (4)\\ &= 2\pi r\sqrt{\frac{r-3GM}{GM}} & (5) \end{aligned}$$

Defining the velocity as $2\pi r/T$, we get

$$v = \sqrt{\frac{GM}{r-3GM}} \tag{6}$$

The period and velocity as measured at infinity are found from the angular speed at infinity Ω

$$\begin{aligned}
\Omega &= \frac{\sqrt{GM}}{r^{3/2}} && (7)\\
T_{\infty} &= \frac{2\pi}{\Omega} && (8)\\
&= 2\pi r\sqrt{\frac{r}{GM}} && (9)\\
v_{\infty} &= \sqrt{\frac{GM}{r}} && (10)
\end{aligned}$$

Finally, the period T_0 as measured by the observer at rest at r is found from the relation between τ and t, as before.

$$\Delta\tau = \sqrt{1-\frac{2GM}{r}}\Delta t \tag{11}$$

In this case, $\Delta t = T_{\infty}$ and $\Delta\tau = T_0$ so

$$\begin{aligned}
T_0 &= 2\pi r\sqrt{1-\frac{2GM}{r}}\sqrt{\frac{r}{GM}} && (12)\\
&= 2\pi r\sqrt{\frac{r-2GM}{GM}} && (13)\\
v_0 &= \sqrt{\frac{GM}{r-2GM}} && (14)
\end{aligned}$$

Thus $T < T_0 < T_{\infty}$ and $v > v_0 > v_{\infty}$. The condition $T < T_0$ could be explained by time dilation, since to the observer at rest, the orbiting clock would run slow, so less time would appear to elapse on it than on the stationary clock. The condition $T_0 < T_{\infty}$ is a consequence of the difference between the proper time for an observer at rest at a finite distance from the mass and the time at infinity, which is given by the time coordinate t.

Since the square roots must all be real, we must have $r \geq 3GM$. This ensures that $v_{\infty} < v_0 \leq 1$, but for $3GM < r < 4GM$, $v > 1$. I'm not entirely sure what the resolution of this apparent paradox is, but in the frame of the orbiting object, its own velocity is, of course, zero, so at best the expression for v above must be considered an artificial velocity which cannot be measured in any physical sense. If the speed of the object is measured by an external observer (such as the stationary observers at r and infinity) the value always seems to be less than 1.

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