PHOTON ORBITS: SPEED MEASURED AT TWO PLACES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problem 10.13.

As we'll see later, a photon can orbit a mass at a radius of $r=3GM$ (in practice, this means that the mass must be very dense, such as a black hole, in order for such a small radius to be outside the mass). Since a photon's world line must always have $ds^2=0$, we can use the Schwarzschild metric to find the speed V of a photon in such an orbit, as measured by an observer at infinity, where the time is measured by the coordinate t.

$$
\begin{aligned}
ds^2 &= 0=-\left(1-\frac{2GM}{r}\right)dt^2+r^2d\phi^2 && (1)\\
&= -\frac{1}{3}dt^2+9\left(GM\right)^2d\phi^2 && (2)\\
\frac{d\phi}{dt} &= \frac{1}{\sqrt{27}GM} && (3)
\end{aligned}
$$

If we define $V\equiv r\frac{d\phi}{dt}$, then

$$V=\frac{3GM}{\sqrt{27}GM}=\frac{1}{\sqrt{3}}=0.577 \tag{4}$$

For a stationary observer at radius $r=3GM$, we can apply the transformation from Δt to $\Delta\tau$ to get

$$
\begin{aligned}
\frac{d\phi}{d\tau} &= \left(1-\frac{2GM}{r}\right)^{-1/2}\frac{d\phi}{dt} && (5)\\
&= \sqrt{3}\frac{d\phi}{dt} && (6)
\end{aligned}
$$

Thus the speed of light as measured by this observer is

$$v=r\frac{d\phi}{d\tau}=\sqrt{3}r\frac{d\phi}{dt}=\sqrt{3}V=1 \tag{7}$$

The difference is due to the fact that both observers are using the same physical distance r, but disagree on the time scale to use. The observer at $3GM$ uses the same time coordinate as the photon, so it measures the photon's speed as 1, but to the observer at infinity, more time appears to have passed while the photon traverses the same distance, so the speed is less.