ORBIT OF A COMET AROUND A BLACK HOLE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; Problem 10.14.

We'll have a look at a non-circular orbit in this post, by considering a comet that falls towards a mass (such as a black hole) from a large (essentially infinite) distance, but which has non-zero angular momentum. This means that the comet will swing past the black hole and then recede back out to infinity. We'll be concerned here only with the point of closest approach, which we'll call R.

The comet starts off essentially at rest, but with an infinitesimal angular momentum. As we saw in a previous post, this implies that the angular speed $\omega = d\phi/d\tau \to 0$ as $r \to \infty$ and the energy per unit mass $e = 1$.

From the definitions of l and e we have, since we're working in an equatorial plane where $\theta = \pi/2$

$$\frac{d\phi}{d\tau} = \frac{l}{r^2 \sin^2\theta} = \frac{l}{r^2} \tag{1}$$

$$e = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \tag{2}$$

These equations are valid at all distances r, so in general, we have, since $e = 1$

$$\frac{d\phi}{d\tau}e = \frac{l}{r^2}\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \tag{3}$$

$$\frac{d\phi}{d\tau} = \frac{l}{r^2}\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \tag{4}$$

$$u^\phi = \frac{l}{r^2}\left(1 - \frac{2GM}{r}\right)u^t \tag{5}$$

Now we'll restrict ourselves to the point of closest approach, where $r = R$. At this point, the radius is at a minimum, so $dr/d\tau = 0$. The four-velocity of the comet will therefore have only its t and ϕ components non-zero, so, using the Schwarzschild metric and the condition $\mathbf{u}\cdot\mathbf{u} = -1$:

$$-\left(1-\frac{2GM}{R}\right)\left(u^t\right)^2+R^2\left(u^\phi\right)^2=-1 \tag{6}$$

$$\left(u^t\right)^2\left[-\left(1-\frac{2GM}{R}\right)+\frac{R^2l^2}{R^4}\left(1-\frac{2GM}{R}\right)^2\right]=-1 \tag{7}$$

$$u^t=\frac{1}{\sqrt{\left(1-\frac{2GM}{R}\right)-\frac{l^2}{R^2}\left(1-\frac{2GM}{R}\right)^2}} \tag{8}$$

$$p^t=\frac{m}{\sqrt{\left(1-\frac{2GM}{R}\right)-\frac{l^2}{R^2}\left(1-\frac{2GM}{R}\right)^2}} \tag{9}$$

Now we can use the invariant equation $E_{obs}=-\mathbf{p}\cdot\mathbf{u}_{obs}$ to find the energy of the comet as seen by a stationary observer also at distance R. The observer's four-velocity is

$$\mathbf{u}_{obs}=\left[\left(1-\frac{2GM}{R}\right)^{-1/2},0,0,0\right] \tag{10}$$

so we get

$$\begin{aligned} E_{obs} &= \left(1-\frac{2GM}{R}\right)p^t u^t_{obs} && (11)\\ &= \left(1-\frac{2GM}{R}\right)^{1/2}\frac{m}{\sqrt{\left(1-\frac{2GM}{R}\right)-\frac{l^2}{R^2}\left(1-\frac{2GM}{R}\right)^2}} && (12)\\ &= \frac{m}{\sqrt{1-\frac{l^2}{R^2}\left(1-\frac{2GM}{R}\right)}} && (13)\end{aligned}$$

From $E=m/\sqrt{1-v^2}$ we get the speed of the comet at its closest approach:

$$v=\frac{l}{R}\sqrt{1-\frac{2GM}{R}} \tag{14}$$

We still haven't found R however. We can do this from the radial equation of motion:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+\frac{1}{2}\frac{l^2}{R^2}-GM\left(\frac{1}{R}+\frac{l^2}{R^3}\right)=\frac{1}{2}\left(e^2-1\right) \tag{15}$$

Using $dr/d\tau = 0$ and $e = 1$, we get

$$\frac{1}{2}\frac{l^2}{R^2} - GM\left(\frac{1}{R} + \frac{l^2}{R^3}\right) = 0 \tag{16}$$

Multiplying through by R^3 gives a quadratic equation which has roots

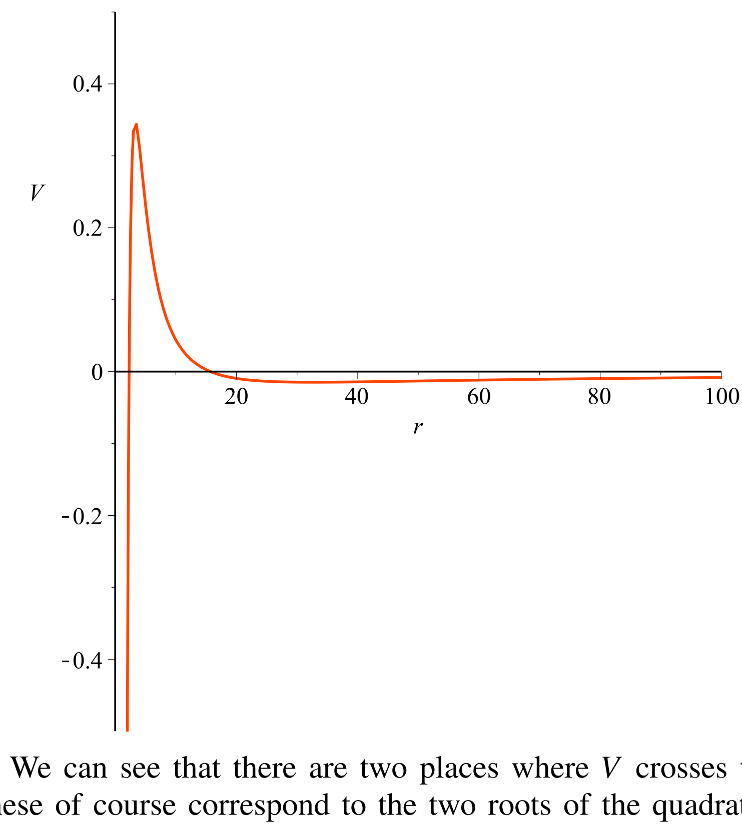
$$R = \frac{l^2}{4GM} \pm \frac{1}{2}\sqrt{\frac{l^4}{(2GM)^2} - 4l^2} \tag{17}$$

$$= \frac{l^2}{4GM}\left[1 \pm \sqrt{1 - \frac{(4GM)^2}{l^2}}\right] \tag{18}$$

To analyze these two solutions, we can look at a plot of

$$V \equiv \frac{1}{2}\frac{l^2}{r^2} - GM\left(\frac{1}{r} + \frac{l^2}{r^3}\right) \tag{19}$$

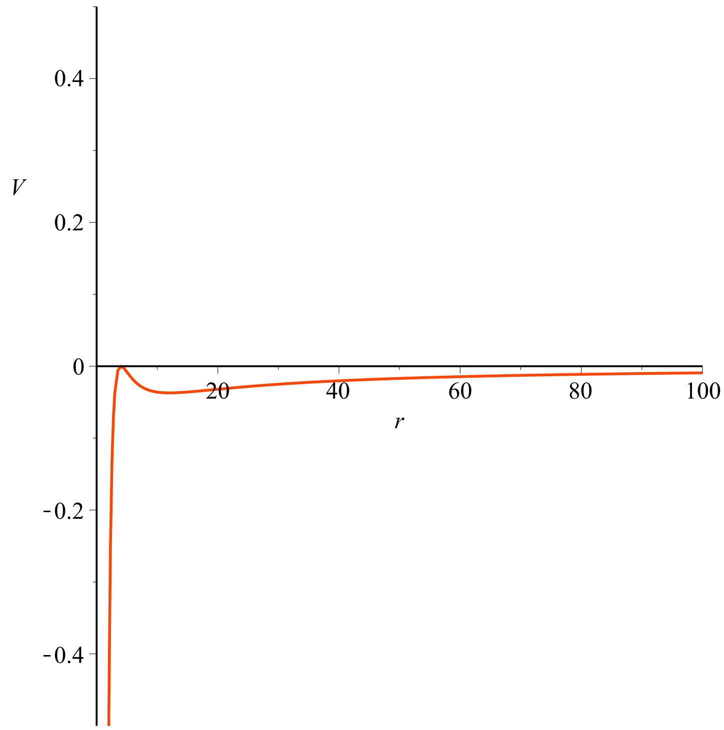
Here we show this for r in units of GM and $l = 6GM$.



We can see that there are two places where V crosses the r axis, and these of course correspond to the two roots of the quadratic. As we are approaching from $r = \infty$, the outer root is the one we encounter first, after which the comet reverses direction and r increases again. Thus the required solution is the larger of the two, which is

$$R=\frac{l^{2}}{4GM}\left[1+\sqrt{1-\frac{(4GM)^{2}}{l^{2}}}\right] \tag{20}$$

The minimum value of l for which a solution exists is $l=4GM$, which gives rise to a single solution as we can see from the plot:



For large l, the second term inside the square root can be neglected, and we get

$$R\to\frac{l^{2}}{2GM} \tag{21}$$

This corresponds to the Newtonian case, since the energy in that case is

$$E=\frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^{2}+\left(r\frac{d\phi}{dt}\right)^{2}\right)-\frac{GMm}{r} \tag{22}$$

The angular momentum per unit mass is $l=r^{2}\frac{d\phi}{dt}$ and at closest approach $\frac{dr}{dt}=0$. For a comet that starts off at rest at infinity, $E=0$, so

$$0 = \frac{1}{2}\frac{l^2}{R^2} - \frac{GM}{R} \tag{23}$$

$$R = \frac{l^2}{2GM} \tag{24}$$

Thus the Schwarzschild case reduces to the Newtonian case for large l. From 20, we can see that the Schwarzschild minimum distance is always smaller than the Newtonian one. This makes sense, since if we compare the Newtonian and Schwarzschild equations of motion for unit mass:

$$E_N = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{l^2}{r^2} - \frac{GM}{r} \equiv \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V_N \tag{25}$$

$$E_S = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{l^2}{r^2} - GM\left(\frac{1}{r} + \frac{l^2}{r^3}\right) \equiv \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V_S \tag{26}$$

we can see that there is an extra negative term in V_S so we would expect V_S to have its zero at smaller values of r.