

TWIN PARADOX WITH A BLACK HOLE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 10; 15.

The twin paradox in special relativity involves one twin remaining on Earth while the other twin takes off on a trip to some distant star at relativistic speeds. Due to time dilation, the travelling twin's clock runs slow compared to the stationary twin, so that when the traveller returns, he is younger than his brother.

Another type of twin paradox occurs in general relativity, in which the travelling twin makes a trip to a black hole, does some orbits around it, and then returns. Because time travels more slowly near a massive object, the traveller is again younger than his brother.

As a first example, we recall the example of an object starting off at rest except for a slight tangential velocity. We found that if the angular momentum per unit mass is $l = 4GM$, then the object can spiral in to an unstable circular orbit at $r = 4GM$. We need to calculate the time dilation effect for this orbit.

We've already done the essentials. We've seen that the period as measured by the orbiting object is

$$T = 2\pi r \sqrt{\frac{r - 3GM}{GM}} \quad (1)$$

The period as measured by an observer at infinity is

$$T_\infty = 2\pi r \sqrt{\frac{r}{GM}} \quad (2)$$

Therefore, the time dilation factor is

$$\frac{T}{T_\infty} = \sqrt{\frac{r - 3GM}{r}} \quad (3)$$

$$= \frac{1}{2} \quad (4)$$

Thus for $r = 4GM$, time passes twice as fast for the distant observer.

Now let's invert the problem and specify the required time dilation factor first. Suppose we want $T/T_\infty = 0.1$. This leads to a radius of

$$0.1 = \sqrt{\frac{r-3GM}{r}} \quad (5)$$

$$r = \frac{3}{0.99}GM \quad (6)$$

From this we can get the angular momentum per unit mass:

$$l^2 = \frac{r^2GM}{r-3GM} \quad (7)$$

Now we can use the radial equation of motion:

$$\tilde{E} = \frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 + \frac{1}{2} \frac{l^2}{r^2} - GM \left(\frac{1}{r} + \frac{l^2}{r^3} \right) \quad (8)$$

In this case, we can't assume that the total energy is zero, since we're not starting off with a stationary object at infinity. However, we do know that when the object is in the circular orbit $dr/d\tau = 0$, so

$$\tilde{E}(r_c) = \frac{1}{2} \frac{l^2}{r_c^2} - GM \left(\frac{1}{r_c} + \frac{l^2}{r_c^3} \right) \quad (9)$$

$$= \frac{GM}{2(r_c-3GM)} - GM \left(\frac{1}{r_c} + \frac{GM}{r_c(r_c-3GM)} \right) \quad (10)$$

$$= \frac{GM(4GM-r_c)}{2r_c(r_c-3GM)} \quad (11)$$

At infinity, the same energy must hold, so we have, where r_c is the radius of the circular orbit:

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 = \frac{GM(4GM-r_c)}{2r_c(r_c-3GM)} \quad (12)$$

$$\frac{dr}{d\tau} = \sqrt{\frac{GM(4GM-r_c)}{r_c(r_c-3GM)}} \quad (13)$$

However, since the object is moving, its proper time τ is not the same as the clock time t for an observer at rest at infinity. To get the transformation, we can use the invariant relation $\mathbf{u} \cdot \mathbf{u} = -1$, and the relation

$$e = \left(1 - \frac{2GM}{r} \right) \frac{dt}{d\tau} \quad (14)$$

At infinity, $e = \frac{dt}{d\tau}$ and, since $\tilde{E} = \frac{1}{2}(e^2 - 1)$, we have

$$\tilde{E}(r_c) = \frac{1}{2} \left(\left(\frac{dt}{d\tau} \right)^2 - 1 \right) = \frac{GM(4GM - r_c)}{2r_c(r_c - 3GM)} \quad (15)$$

$$\left(\frac{dt}{d\tau} \right)^2 = 1 + \frac{GM(4GM - r_c)}{r_c(r_c - 3GM)} \quad (16)$$

$$= \frac{(r_c - 2GM)^2}{r_c(r_c - 3GM)} \quad (17)$$

At infinity, $g_{tt} = -1$ and $g_{rr} = 1$ and since the only motion is in the radial direction (the tangential velocity is infinitesimal), we have

$$\mathbf{u} \cdot \mathbf{u} = - \left(\frac{dt}{d\tau} \right)^2 + \left(\frac{dr}{d\tau} \right)^2 = -1 \quad (18)$$

$$\left(\frac{dr}{d\tau} \right)^2 \left(\frac{d\tau}{dt} \right)^2 = - \left(\frac{d\tau}{dt} \right)^2 + 1 \quad (19)$$

$$\left(\frac{dr}{dt} \right)^2 = 1 - \frac{r_c(r_c - 3GM)}{(r_c - 2GM)^2} \quad (20)$$

$$= \frac{GM(4GM - r_c)}{(r_c - 2GM)^2} \quad (21)$$

The initial radial velocity as measured by an observer at rest at infinity is therefore

$$\frac{dr}{dt} = \frac{\sqrt{GM(4GM - r_c)}}{r_c - 2GM} \quad (22)$$

For the particular example here, $r_c = \frac{3}{0.99}GM$ so

$$\frac{dr}{dt} = 0.9558 \quad (23)$$

If we travelled at this speed in flat space, the time dilation factor is

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = 3.4 \quad (24)$$

so the gravitational time dilation factor of 10 that we specified initially is about 3 times that in flat space.