PERIHELION SHIFT - CONTRIBUTION FROM THE RADIAL COORDINATE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 11; Exercise 11.6.1.

In analyzing the precession of an object's closest approach in its orbit around a central mass we found that, for nearly circular orbits where the mean radius r_c of the orbit is much greater than GM (M is the central mass), the angle of closest approach advances by

$$\Delta\phi = \frac{6\pi GM}{r_c} \tag{1}$$

on each orbit.

Since the only two components in the Schwarzschild metric that differ from flat space are the radial and time components, this shift must be due to contributions from each of these coordinates. To see how much arises from each coordinate, we can hold the other one constant and see what perihelion shift arises.

First, consider the radial coordinate r. In this case, $dt = d\theta = 0$ (the latter because we're moving in the equatorial plane, as usual), so the metric becomes

$$ds^2 = \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\phi \tag{2}$$

This is a 2-d curved space but, just as we can visualize the 2-d curved space of a sphere by embedding it in 3-d flat space, we can do the same here. Since the metric is independent of ϕ, the natural coordinate system to use in the flat space is the cylindrical system, with coordinates z, r and ϕ. In the cylindrical system

$$ds^2 = dr^2 + r^2 d\phi + dz^2 \tag{3}$$

so we can equate the two metrics to get

$$\left(1-\frac{2GM}{r}\right)^{-1}dr^2+r^2d\phi = dr^2+r^2d\phi+dz^2 \tag{4}$$

$$\frac{dz}{dr} = \sqrt{\left(1-\frac{2GM}{r}\right)^{-1}-1} \tag{5}$$

$$= \sqrt{\frac{2GM/r}{1-2GM/r}} \tag{6}$$

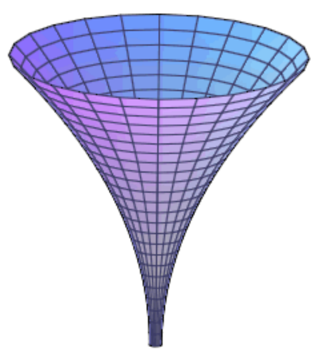
$$= \frac{\sqrt{2GM}}{\sqrt{r-2GM}} \tag{7}$$

This is quite easy to integrate, and we get

$$z = 2\sqrt{2GM\left(r-2GM\right)} \tag{8}$$

$$= \sqrt{8GM\left(r-2GM\right)} \tag{9}$$

The 3-d plot of this surface looks like this:



The surface is known as *Flamm's paraboloid* (Try as I might, I couldn't find out who Flamm is or was. There was an Austrian table tennis player by that name, but it's unlikely to be her.) and has become something of an icon in discussions of black holes, since it seems to indicate that there is a 'gravity well' around the black hole into which objects fall. This is misleading, since the paraboloid is a snapshot of the space at one particular instant in time (remember we took $dt = 0$) so an object near a black hole does *not* move along the surface of the paraboloid. We'll look at this in more detail later.

To use this paraboloid to determine the amount of the perihelion shift due to the radial component, we can start by looking at an orbit in flat space. Suppose we consider a nearly circular orbit of radius R. In flat space, we can draw this as a nearly-circular ellipse on a flat sheet of paper. Also, in the absence of a central mass, there is no perihelion shift, so the object goes through exactly 2π radians from one perihelion to the next.

Now if we introduce a central mass, the space becomes curved into the shape of Flamm's paraboloid. For $R \gg GM$ (our usual assumption), we can approximate the curvature of space by imagining that the plane in flat space is deformed into a cone that is tangent to the paraboloid. That is, if you imagine the paraboloid as a funnel, we put a conical filter paper into the funnel, and this paper touches the funnel in an almost-circle. As you may have done in high school chemistry, you can make a cone out of a circular piece of paper by cutting the circle from its centre along a radius out to the edge, then overlapping the cut edges by a certain amount. The angle δ subtended by the overlapping portions of the cut circle determine how steep the sides of the cone are.

To introduce some quantities, we know that the distance from the vertex of the cone to the rim is just the mean radius R of the original ellipse. Now suppose that this line from vertex to rim makes an angle α with a radius r on the base of the cone. That is, we draw a line from the centre of the base along a radius out to the edge of the base, then from that point up the side of the cone to the vertex. The angle α is the angle between the two lines where they meet at the rim of the base. The radius of the base is then $r = R\cos\alpha$ and the circumference of the base is $2\pi r = 2\pi R\cos\alpha$.

However, the circumference of the base is also the circumference of the original circle minus the overlapping bit. That is, we must also have

$$2\pi r = 2\pi R\cos\alpha = (2\pi - \delta)R \tag{10}$$

$$2\pi\cos\alpha = (2\pi - \delta) \tag{11}$$

For $R \gg GM$, the angle α will be very small, so $\cos\alpha \approx 1 - \frac{1}{2}\alpha^2$. Also, $\tan\alpha$ is the slope of the paraboloid at the point where the cone is tangent to it, so

$$\tan\alpha = \left.\frac{dz}{dr}\right|_{r=R\cos\alpha} \tag{12}$$

$$= \frac{\sqrt{2GM}}{\sqrt{R\cos\alpha - 2GM}} \tag{13}$$

$$\approx \frac{\sqrt{2GM}}{\sqrt{R\left(1-\frac{1}{2}\alpha^2\right) - 2GM}} \tag{14}$$

$$\approx \alpha \tag{15}$$

We can now square both sides to get a quadratic equation for α^2:

$$\frac{R}{2}\alpha^4 - (R-2GM)\,\alpha^2 + 2GM = 0 \tag{16}$$

$$\alpha^2 = \frac{1}{R}\left[R - 2GM \pm \sqrt{(R-2GM)^2 - 4RGM}\right] \tag{17}$$

$$= \frac{1}{R}(R-2GM)\left[1 \pm \sqrt{1 - \frac{4RGM}{(R-2GM)^2}}\right] \tag{18}$$

$$\approx \left(1 - \frac{2GM}{R}\right)\left[1 \pm \left(1 - \frac{2RGM}{(R-2GM)^2}\right)\right] \tag{19}$$

$$\approx \left(1 - \frac{2GM}{R}\right)\left(1 \pm \left(1 - \frac{2GM}{R}\right)\right) \tag{20}$$

where in the last two lines we used the condition $R \gg GM$. Since α is small, we need to take the minus sign in the last line, and saving up to first-order terms in $2GM/R$ we get

$$\alpha^2 \approx \frac{2GM}{R} \tag{21}$$

Plugging this back into the equation above for the overlap angle δ (which is the perihelion shift due to the radial coordinate), we find

$$2\pi\left(1 - \frac{1}{2}\alpha^2\right) = 2\pi - \delta \tag{22}$$

$$\delta = \frac{2\pi GM}{R} \tag{23}$$

This is $\frac{1}{3}$ of the total perihelion shift as given at the start.

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