

EMBEDDING 2-D CURVED SPACE IN 3-D: THE SPHERE

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 11; Problem 11.06.

We can now look at some examples of embedding a curved-space 2-d metric in 3-d flat space. We'll begin with the familiar case of the spherical surface. However, the goal here is to start with a metric defined in terms of some unknown coordinates and then discover the nature of the surface by embedding it in 3-d space.

The metric is, not surprisingly

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2 \quad (1)$$

To get this in the form where we can embed it using cylindrical coordinates, we need the ϕ term to be $r^2 d\phi^2$ for some coordinate r . Therefore, we can try just defining $r = R \sin \theta$ and see what this does to the θ component. We get

$$dr = R \cos \theta d\theta \quad (2)$$

$$d\theta = \frac{dr}{R \cos \theta} \quad (3)$$

$$= \frac{dr}{R \sqrt{1 - \sin^2 \theta}} \quad (4)$$

$$= \frac{dr}{R \sqrt{1 - \left(\frac{r}{R}\right)^2}} \quad (5)$$

Thus the metric can be rewritten in terms of r and ϕ as follows:

$$ds^2 = \frac{dr^2}{1 - \left(\frac{r}{R}\right)^2} + r^2 d\phi^2 \quad (6)$$

From here, we equate this to the 3-d cylindrical metric:

$$dz^2 + r^2 d\phi^2 + dr^2 = \frac{dr^2}{1 - \left(\frac{r}{R}\right)^2} + r^2 d\phi^2 \quad (7)$$

$$\frac{dz}{dr} = \pm \frac{r/R}{\sqrt{1 - \left(\frac{r}{R}\right)^2}} \quad (8)$$

$$z = \pm R \sqrt{1 - \left(\frac{r}{R}\right)^2} \quad (9)$$

$$= \pm \sqrt{R^2 - r^2} \quad (10)$$

This is the equation of the two halves (upper and lower) of a sphere, so we see that the embedding of the 2-d metric in 3-d space does indeed give a sphere.