

EMBEDDING 2-D CURVED SPACE IN 3-D: THE SPHERE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 11; Problem 11.06.

We can now look at some examples of embedding a curved-space 2-d metric in 3-d flat space. We'll begin with the familiar case of the spherical surface. However, the goal here is to start with a metric defined in terms of some unknown coordinates and then discover the nature of the surface by embedding it in 3-d space.

The metric is, not surprisingly

$$(0.1) \quad ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2$$

To get this in the form where we can embed it using cylindrical coordinates, we need the ϕ term to be $r^2 d\phi^2$ for some coordinate r . Therefore, we can try just defining $r = R \sin \theta$ and see what this does to the θ component. We get

$$(0.2) \quad dr = R \cos \theta d\theta$$

$$(0.3) \quad d\theta = \frac{dr}{R \cos \theta}$$

$$(0.4) \quad = \frac{dr}{R \sqrt{1 - \sin^2 \theta}}$$

$$(0.5) \quad = \frac{dr}{R \sqrt{1 - \left(\frac{r}{R}\right)^2}}$$

Thus the metric can be rewritten in terms of r and ϕ as follows:

$$(0.6) \quad ds^2 = \frac{dr^2}{1 - \left(\frac{r}{R}\right)^2} + r^2 d\phi^2$$

From here, we equate this to the 3-d cylindrical metric:

$$(0.7) \quad dz^2 + r^2 d\phi^2 + dr^2 = \frac{dr^2}{1 - \left(\frac{r}{R}\right)^2} + r^2 d\phi^2$$

$$(0.8) \quad \frac{dz}{dr} = \pm \frac{r/R}{\sqrt{1 - \left(\frac{r}{R}\right)^2}}$$

$$(0.9) \quad z = \pm R \sqrt{1 - \left(\frac{r}{R}\right)^2}$$

$$(0.10) \quad = \pm \sqrt{R^2 - r^2}$$

This is the equation of the two halves (upper and lower) of a sphere, so we see that the embedding of the 2-d metric in 3-d space does indeed give a sphere.