EMBEDDING 2-D CURVED SPACE IN 3-D: THE SPHERE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 11; Problem 11.06.

We can now look at some examples of embedding a curved-space 2-d metric in 3-d flat space. We'll begin with the familiar case of the spherical surface. However, the goal here is to start with a metric defined in terms of some unknown coordinates and then discover the nature of the surface by embedding it in 3-d space.

The metric is, not surprisingly

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \tag{1}$$

To get this in the form where we can embed it using cylindrical coordinates, we need the ϕ term to be $r^2 d\phi^2$ for some coordinate r. Therefore, we can try just defining $r = R\sin\theta$ and see what this does to the θ component. We get

$$\begin{aligned}
dr &= R\cos\theta d\theta && (2)\\
d\theta &= \frac{dr}{R\cos\theta} && (3)\\
&= \frac{dr}{R\sqrt{1-\sin^2\theta}} && (4)\\
&= \frac{dr}{R\sqrt{1-\left(\frac{r}{R}\right)^2}} && (5)
\end{aligned}$$

Thus the metric can be rewritten in terms of r and ϕ as follows:

$$ds^2 = \frac{dr^2}{1-\left(\frac{r}{R}\right)^2} + r^2 d\phi^2 \tag{6}$$

From here, we equate this to the 3-d cylindrical metric:

$$dz^2 + r^2 d\phi^2 + dr^2 = \frac{dr^2}{1-\left(\frac{r}{R}\right)^2} + r^2 d\phi^2 \tag{7}$$

$$\frac{dz}{dr} = \pm\frac{r/R}{\sqrt{1-\left(\frac{r}{R}\right)^2}} \tag{8}$$

$$z = \pm R\sqrt{1-\left(\frac{r}{R}\right)^2} \tag{9}$$

$$= \pm\sqrt{R^2-r^2} \tag{10}$$

This is the equation of the two halves (upper and lower) of a sphere, so we see that the embedding of the 2-d metric in 3-d space does indeed give a sphere.