EMBEDDING A 2-D CURVED SURFACE IN 3-D: THE COSINE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 11; Problem 11.08.

A third example embedding a curved-space2-d metric in 3-d flat space. This time, our 2-d metric is

$$ds^2 = \frac{dr^2}{\cos^2(r/R)} + r^2 d\phi^2 \tag{1}$$

Here, the ϕ component is already in the required form, so we can equate this to the cylindrical metric directly:

$$\begin{aligned} dz^2 + r^2 d\phi^2 + dr^2 &= \frac{dr^2}{\cos^2(r/R)} + r^2 d\phi^2 && (2) \\ \frac{dz}{dr} &= \pm\sqrt{\frac{1-\cos^2(r/R)}{\cos^2(r/R)}} && (3) \\ &= \pm\tan\left(\frac{r}{R}\right) && (4) \end{aligned}$$

This integrates directly to give

$$z = \pm R \ln\left|\cos\left(\frac{r}{R}\right)\right| \tag{5}$$

This integral is a bit problematic, as the logarithm is defined only for positive arguments, which is why we've put the absolute value in the answer. If the limits include a region where the cosine is zero, the log goes to infinity, so in our case here, $0 \le r/R < \pi/2$. (This also follows from the original metric, since the cosine is in the denominator.) Since the cosine is always ≤ 1, the log is always negative.

The lobe of this surface obtained from taking the + sign above looks like this (there is an upper lobe which is a mirror image of the lower one):