EMBEDDING A 2-D CURVED SURFACE INTO 3-D: INVERSE COSH

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 11; Problem 11.09.

A final example embedding a curved-space 2-d metric in 3-d flat space. This time, our 2-d metric is

$$ds^2 = d\rho^2 + \left(\rho^2 + b^2\right) d\phi^2 \tag{0.1}$$

where b is a constant.

To convert the ϕ component to the required form of $r^2 d\phi^2$, we define $r = \sqrt{\rho^2 + b^2}$. Then

$$\begin{aligned} dr &= \frac{\rho}{\sqrt{\rho^2+b^2}} d\rho && (0.2) \\ &= \frac{\sqrt{r^2-b^2}\,d\rho}{r} && (0.3) \\ d\rho &= \frac{r}{\sqrt{r^2-b^2}} dr && (0.4) \end{aligned}$$

The metric becomes:

$$ds^2 = \frac{r^2 dr^2}{r^2 - b^2} + r^2 d\phi^2 \tag{0.5}$$

We now equate this to the cylindrical metric:

$$\begin{aligned} dz^2 + r^2 d\phi^2 + dr^2 &= \frac{r^2 dr^2}{r^2-b^2} + r^2 d\phi^2 && (0.6) \\ \frac{dz}{dr} &= \frac{b}{\sqrt{r^2-b^2}} && (0.7) \end{aligned}$$

This integral can be done with software or looked up, and is

$$z = b\ln\left(r+\sqrt{r^2-b^2}\right) \tag{0.8}$$

$$= b\ln\left[b\left(\frac{r}{b}+\sqrt{\frac{r^2}{b^2}-1}\right)\right] \tag{0.9}$$

$$= b\ln\left(\frac{r}{b}+\sqrt{\frac{r^2}{b^2}-1}\right)+b\ln b \tag{0.10}$$

From tables of inverse hyperbolic functions, we see that this is equivalent to

$$z = b\operatorname{arcosh}\left(\frac{r}{b}\right)+b\ln b \tag{0.11}$$

We can ignore the last term as it is just a constant and serves only to raise or lower the surface as a whole.

The surface is similar to Flamm's paraboloid, although since it is derived from hyperbolic functions and not parabolas, it's not a paraboloid.