PERIHELION SHIFT: NUMERICAL SOLUTION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 11; Problem 11.11.

The general case of an object orbiting a much larger mass is treated by the equations for radius and angle as functions of proper time:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+\frac{1}{2}\frac{l^2}{r^2}-GM\left(\frac{1}{r}+\frac{l^2}{r^3}\right) = \frac{1}{2}\left(e^2-1\right) \tag{1}$$

$$r^2\frac{d\phi}{d\tau} = l \tag{2}$$

Taking the derivative of the first equation with respect to τ and cancelling off the common factor of $dr/d\tau$ that results, gives us

$$\frac{d^2r}{d\tau^2}=-\frac{GM}{r^2}+\frac{l^2}{r^3}-\frac{3GMl^2}{r^4} \tag{3}$$

This equation, together with the one for ϕ above, make up a coupled system of two ODEs. In general, there is no analytic solution to them, but they can be solved numerically to give parametric equations for r and ϕ as functions of τ, which can then be combined to give $r(\phi)$ which, when plotted, gives us a diagram of the orbit.

Since the r equation is second order, we need to specify r and $dr/d\tau$ at $\tau=0$, while for the first order ϕ equation, we need to specify only $\phi(0)$. We can also specify $d\phi/d\tau$ at $\tau=0$ and use this, together with $r(0)$ to determine $l=r^2(0)\,\phi'(0)$. To specify a value for $\phi'(0)$, we can take a certain fraction ϕ_0 of the angular speed ϕ' for a circular orbit of radius $r(0)$. For a circular orbit

$$l^2=r^4\left(\phi'\right)^2 = \frac{r^2GM}{r-3GM} \tag{4}$$

$$\phi' = \frac{\sqrt{GM}}{r\sqrt{r-3GM}}=\frac{1}{r\sqrt{r-3}} \tag{5}$$

where at the end, we've written r in units of GM. Thus taking a fraction ϕ_0 of this angular speed gives

$$l = r^2(0)\,\phi'(0) \tag{6}$$

$$= \phi_0 \frac{r(0)}{\sqrt{r(0)-3}} \tag{7}$$

If we write r and l in units of GM, then the equations become

$$r^2 \frac{d\phi}{d\tau} = l \tag{8}$$

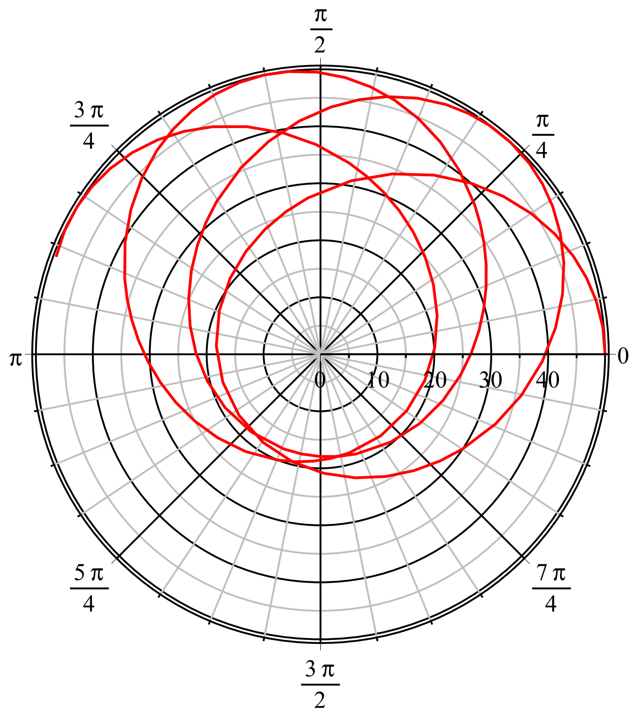
$$\frac{d^2 r}{d\tau^2} = -\frac{1}{r^2} + \frac{l^2}{r^3} - \frac{3l^2}{r^4} \tag{9}$$

There are various ways of integrating these equations numerically, but if you have access to mathematical software such as Maple or Mathematica, the easiest way is to get them to do the hard work for you. A maple procedure that solves the equations and then draws a plot of the result is as follows:

```
[code]
path := proc (r0, rp0, phi0, tup)
  local sys, l, sol, R, Phi;
  l := r0*phi0/sqrt(r0-3);
  sys := {phi(0) = 0, r(0) = r0,
    diff(phi(t), t) = l/r(t)^2,
    diff(r(t), `$`(t, 2)) = -1/r(t)^2+l^2/r(t)^3-
      3*l^2/r(t)^4, (D(r))(0) = rp0};
  sol := dsolve(sys, {phi(t), r(t)}, type = numeric,
    output = listprocedure);
  Phi := subs(sol, phi(t));
  R := subs(sol, r(t));
  plot([R(t), Phi(t), t = 0 .. tup], coords = polar,
    axiscoordinates = polar);
end proc
[/code]
```

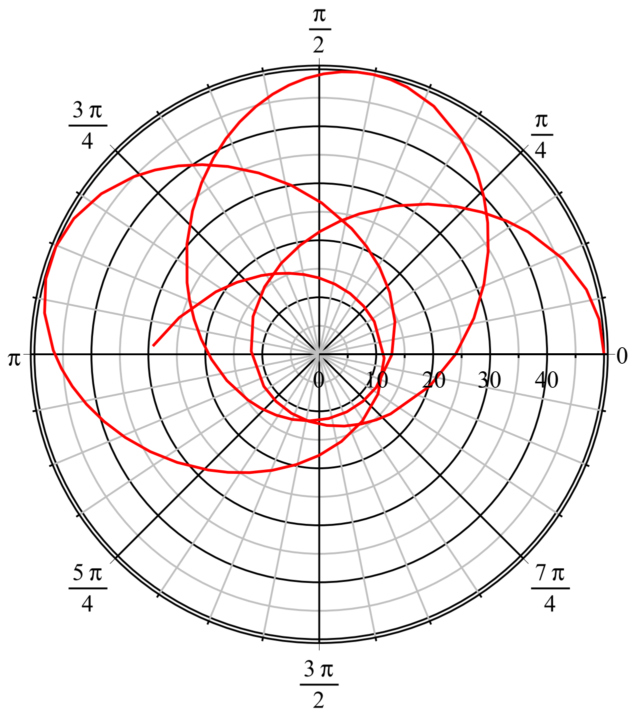
We can choose the starting point at $\tau = 0$ to be a stationary point (where $r' = 0$) and label this angle as $\phi = 0$, so $r'(0) = 0$. These two initial conditions will always be true, so the analysis involves changing $r(0)$ and ϕ_0. Here are a few plots of the results. In each plot, the red curve shows $r(\phi)$, with r in units of GM.

First, we look at $r = 50$ and $\phi_0 = 0.75$. Here are 3 orbits:



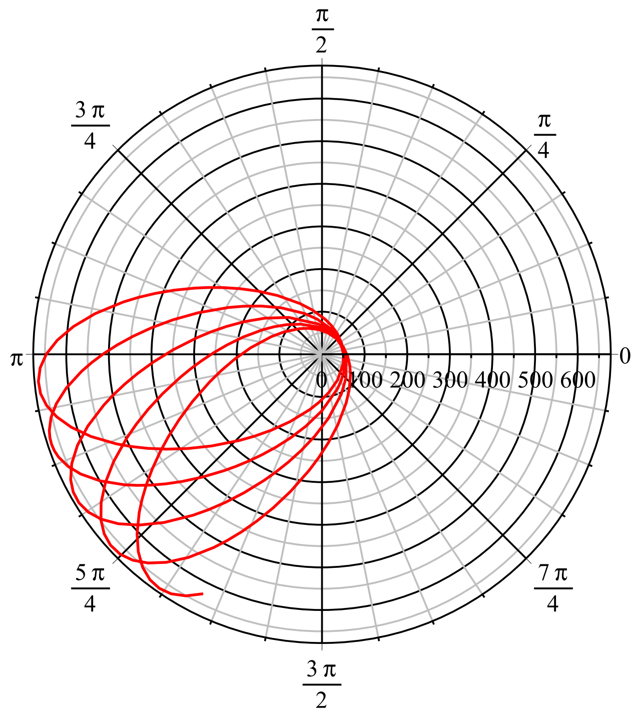
This displays the precession quite nicely. The starting point is an aphelion (greatest distance) point. The perihelion occurs at about $r_{min} \approx 18$.

Now suppose we decrease ϕ_0 to 0.65. The result is:

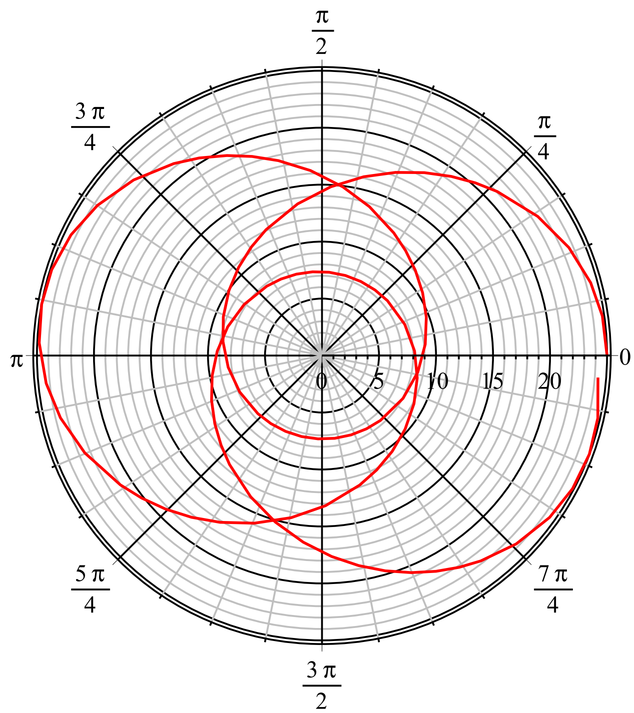


Two effects are noticeable here. First, the object gets closer to the central mass (with $r_{min} \approx 11$). This might be expected, since the object has a smaller tangential velocity so would be pulled further towards the centre. Second, the perihelion shift per orbit is larger. Interpreting this is a bit trickier, since the formula we got earlier for almost-circular orbits ($\Delta\phi = 6\pi GM/r_c$) can't be used for such non-circular orbits. Probably a simple explanation is that as the object gets closer to the centre, it gets a bigger kick from the highly curved space there which distorts its orbit more, causing a larger perihelion shift.

If we decrease ϕ_0 further, we soon come to a point where the object spirals in to $r = 0$. Increasing ϕ_0 from 0.75 makes the orbit more nearly circular with smaller $\Delta\phi$ until, at $\phi_0 = 1$, we get an exactly circular orbit. Increasing ϕ_0 beyond 1 results in the object's starting point becoming its perihelion instead of aphelion, with the aphelion getting further away as ϕ_0 is increased. Here's an example with $\phi_0 = 1.35$:



Changing $r(0)$ has similar effects. Decreasing $r(0)$ causes an increase in perihelion shift, since the object is closer to the centre. This is similar to the effect of decreasing ϕ_0 above. Here's $r(0) = 25$ and $\phi_0 = 0.75$:



$\frac{\pi}{2}$
$\frac{3\pi}{4}$
$\frac{\pi}{4}$
π
0
0
5
10
15
20
$\frac{5\pi}{4}$
$\frac{7\pi}{4}$
$\frac{3\pi}{2}$