LOCAL FLAT COORDINATE SYSTEMS; FOUR-MOMENTUM OF PHOTONS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 12; Problem 12.3.

Sometimes it's useful to use a local coordinate system to do calculations. If the spacetime metric (such as the Schwarzschild metric) is smooth over a region of space (that is, it has no singularities such as division by zero and no sudden jumps), then we can define a local metric that is essentially flat. A simple example is that of the surface of the Earth. Although the Earth is spherical, a locally flat 2-d coordinate system works well for distances of, say, a few miles.

In 4-d spacetime, we can define a locally flat coordinate system with the four mutually orthogonal basis vectors

$$\mathbf{o}_t = [1,0,0,0] \tag{1}$$
$$\mathbf{o}_x = [0,1,0,0] \tag{2}$$
$$\mathbf{o}_y = [0,0,1,0] \tag{3}$$
$$\mathbf{o}_z = [0,0,0,1] \tag{4}$$

and the usual flat space metric from special relativity η_{ij}.

If we know some four vector $\mathbf{A}$ in this flat coordinate system, we can find its components by taking the scalar product with each of the basis vectors. That is

$$\mathbf{o}_x \cdot \mathbf{A} = \eta_{ij} o_x^i A^j \tag{5}$$
$$= \eta_{xx} A^x \tag{6}$$
$$= A^x \tag{7}$$

The second line follows because the only non-zero component of $\mathbf{o}_x$ is o_x^x so $i = x$, and then the only non-zero component of the metric η_{xj} is $\eta_{xx} = +1$. By the same argument, we get the other components:

$$\mathbf{o}_t \cdot \mathbf{A} = -A^t \tag{8}$$
$$\mathbf{o}_y \cdot \mathbf{A} = A^y \tag{9}$$
$$\mathbf{o}_z \cdot \mathbf{A} = A^z \tag{10}$$

This might seem trivial but the important point is that since the scalar product is invariant, if we work out the components in one coordinate system, then if we can find the basis vectors $\mathbf{o}_i$ in another coordinate system, their scalar products with **A** in that coordinate system must yield the same numerical results. This is often an easier way of finding the components of **A** in other coordinate systems (as opposed to using the general tensor transformation formula).

The problem then is to find $\mathbf{o}_i$ in the other coordinate system. If we take this system to be the global system with the Schwarzschild metric, we can work out this transformation. First, we need to align the axes in the two systems. We'll take the x, y and z axes in the local system to be aligned with the ϕ,θ and r axes in the general system. To get started, suppose the observer (who has the local, flat system) is at rest in the general system. Then his four velocity $\mathbf{u}'$ as measured in the general system must have all its spatial components equal to zero. Using the Schwarzschild metric, we must have

$$\mathbf{u}' \cdot \mathbf{u}' = g_{tt} u'^t u'^t \tag{11}$$
$$= -\left(1-\frac{2GM}{r}\right)\left(u'^t\right)^2 \tag{12}$$
$$= -1 \tag{13}$$
$$u'^t = \left(1-\frac{2GM}{r}\right)^{-1/2} \tag{14}$$

In the observer's own local frame, because the metric is flat and the observer is not moving relative to himself, $\mathbf{u} = [1,0,0,0]$. That is, in the local frame, $\mathbf{u} = \mathbf{o}_t$. Therefore, $\mathbf{u}'$ is the transformed version $\mathbf{o}'_t$ of the time basis vector, and

$$\mathbf{o}'_t = \left[\left(1-\frac{2GM}{r}\right)^{-1/2},0,0,0\right] \tag{15}$$

What about transforming the spatial basis vectors? We know that in the flat system $\mathbf{o}_i \cdot \mathbf{o}_j = \eta_{ij}$ so the same must be true in the general system (since these are scalar products). That is, it must also be true that $\mathbf{o}'_i \cdot \mathbf{o}'_j = \eta_{ij}$. If we've aligned the axes in the two systems as stated above, then $\mathbf{o}'_x$ must have zero components along the θ and r directions, so we must have (where the components are listed in the order t, r, θ, ϕ):

$$\mathbf{o}'_x = \left[o'^t_x, 0, 0, o'^\phi_x\right] \tag{16}$$

We must also have

$$\begin{aligned} \mathbf{o}'_t \cdot \mathbf{o}'_x &= g_{tt}\left(1-\frac{2GM}{r}\right)^{-1/2} o'^t_x + g_{\phi\phi}\times 0 \times o'^\phi_x & (17) \\ &= 0 & (18) \end{aligned}$$

from which we get $o'^t_x = 0$. Finally, we must also have

$$\begin{aligned} \mathbf{o}'_x \cdot \mathbf{o}'_x &= g_{\phi\phi}\left(o'^\phi_x\right)^2 = 1 & (19) \\ o'^\phi_x &= \frac{1}{r\sin\theta} & (20) \end{aligned}$$

Therefore

$$\mathbf{o}'_x = \left[0, 0, 0, \frac{1}{r\sin\theta}\right] \tag{21}$$

By the same process, we can work out the other two vectors:

$$\begin{aligned} \mathbf{o}'_y &= \left[0, 0, -\frac{1}{r}, 0\right] & (22) \\ \mathbf{o}'_z &= \left[0, \sqrt{1-\frac{2GM}{r}}, 0, 0\right] & (23) \end{aligned}$$

where the minus sign for $\mathbf{o}'_y$ is because we're taking y to be in the $-\theta$ direction in order to get a right-handed coordinate system.

Having worked out the basis vectors in the general system, if we've calculated the the components of $\mathbf{A}$ in the local, flat system as above, we can use the invariance of the scalar product to work out the components of $\mathbf{A}$ in the general system.

Note that it's *not* true that in the Schwarzschild (or any non-flat) metric, $A'^i = \pm\mathbf{o}'_i \cdot \mathbf{A}'$. For example, suppose that $\mathbf{A}' = [1, 0, 0, 0]$. Then

$$-\mathbf{o}'_t \cdot \mathbf{A}' = -g_{tt}\left(1-\frac{2GM}{r}\right)^{-1/2} A'^t = \left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-1/2} = \left(1-\frac{2GM}{r}\right)^{1/2} \tag{24}$$

That is, the transformations derived above apply only when we start with a locally flat space and then transform to some other metric.

We can now apply this method to the specific four-vector which is **p**, the four-momentum of a photon. As usual, we start with the four-momentum of a particle with rest mass and seek a form that makes no mention of m or τ, the proper time. We get

$$p^i = m\frac{dx^i}{d\tau} \tag{25}$$

$$= m\frac{dx^i}{dt}\frac{dt}{d\tau} \tag{26}$$

$$= me\left(1-\frac{2GM}{r}\right)^{-1}\frac{dx^i}{dt} \tag{27}$$

where we've used the definition of e.

We still need to get rid of m, but when defining e we discovered that it is the energy per unit mass at $r=\infty$, so me is the total energy E of the object at infinity. This is carried over to photons by just defining their four-momentum as

$$p^i = E\left(1-\frac{2GM}{r}\right)^{-1}\frac{dx^i}{dt} \tag{28}$$

For a photon moving in the equatorial plane, we therefore get

$$p^t = E\left(1-\frac{2GM}{r}\right)^{-1}\frac{dt}{dt} = E\left(1-\frac{2GM}{r}\right)^{-1} \tag{29}$$

$$p^r = E\left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{dt} \tag{30}$$

$$= \pm E\left(1-\frac{2GM}{r}\right)^{-1}\left(1-\frac{2GM}{r}\right)\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}} \tag{31}$$

$$= \pm E\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}} \tag{32}$$

$$p^\theta = 0 \tag{33}$$

$$p^\phi = E\left(1-\frac{2GM}{r}\right)^{-1}\frac{d\phi}{dt} \tag{34}$$

$$= E\left(1-\frac{2GM}{r}\right)^{-1}\frac{1}{r^2}\left(1-\frac{2GM}{r}\right)b \tag{35}$$

$$= E\frac{b}{r^2} \tag{36}$$

For p^r and p^ϕ we've used the photon equations of motion.

Now suppose we want the velocity components of the photon as measured by our observer in his local, flat frame. The components are, as measured in the **flat local frame**:

$$v^i = \frac{p^i}{p^t} \tag{37}$$

$$= \frac{\mathbf{o}_i \cdot \mathbf{p}}{-\mathbf{o}_t \cdot \mathbf{p}} \tag{38}$$

Because we've managed to express the components in terms of scalar products, we can evaluate them in any frame, so since we know the components of the $\mathbf{o}_i$s and $\mathbf{p}$ in the general frame, we can do the scalar product in that frame. We must remember to use the Schwarzschild metric in calculating the scalar products in the general frame, of course! So we get (remember $\theta = \frac{\pi}{2}$)

$$-\mathbf{o}_t \cdot \mathbf{p} = -g_{tt}\left(1-\frac{2GM}{r}\right)^{-1/2} E\left(1-\frac{2GM}{r}\right)^{-1} \tag{39}$$

$$= E\left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-1/2}\left(1-\frac{2GM}{r}\right)^{-1} \tag{40}$$

$$= E\left(1-\frac{2GM}{r}\right)^{-1/2} \tag{41}$$

$$\mathbf{o}_x \cdot \mathbf{p} = g_{\phi\phi}\frac{1}{r\sin\theta}E\frac{b}{r^2} \tag{42}$$

$$= r^2\sin^2\theta\frac{1}{r\sin\theta}E\frac{b}{r^2} \tag{43}$$

$$= E\frac{b}{r} \tag{44}$$

$$\mathbf{o}_y \cdot \mathbf{p} = 0 \tag{45}$$

$$\mathbf{o}_z \cdot \mathbf{p} = \pm g_{rr}\left(1-\frac{2GM}{r}\right)^{1/2} E\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}} \tag{46}$$

$$= \pm\left(1-\frac{2GM}{r}\right)^{-1}\left(1-\frac{2GM}{r}\right)^{1/2} E\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}} \tag{47}$$

$$= \pm E\left(1-\frac{2GM}{r}\right)^{-1/2}\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}} \tag{48}$$

We therefore get

$$v_x = \left(1-\frac{2GM}{r}\right)^{1/2}\frac{b}{r} \tag{49}$$

$$v_y = 0 \tag{50}$$

$$v_z = \pm\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}} \tag{51}$$

The magnitude of the velocity is then

$$\sqrt{v_x^2+v_z^2} = \left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}+1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2} \tag{52}$$

$$= 1 \tag{53}$$

Thus the photon's speed is always 1 to the observer in the local frame, which is a relief, since photons must always have speed 1.

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