APPARENT SIZE OF A BLACK HOLE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 12; Problem 12.4.

Suppose we are sitting comfortably at a distance r from a black hole. Since a black hole absorbs light, it will appear as a black patch on the background of stars. However, we know that light travelling near to a gravitating object will spiral into that object if its path is close enough, so a black hole will appear larger than it actually is. We'll find out how much larger here.

It's actually easier to begin with the inverse problem. That is, suppose our observer at r emits a photon at an angle ψ relative to the ray projecting radially outwards from the black hole. If $\psi = 0$, the photon is emitted directly away from the black hole, while if $\psi = \pi$, it is emitted directly towards the black hole. At other angles, the photon will either curve round the black hole and head off to infinity in some direction, or it will be captured by the black hole and spiral in towards it. We want to find the critical angle ψ_c which is the smallest angle at which the photon gets captured.

We've already worked out expressions for the components of the photon's velocity as viewed by the observer. The transverse component is

$$v_x = \left(1 - \frac{2GM}{r}\right)^{1/2} \frac{b}{r} \tag{1}$$

where b is the impact parameter, originally defined as $b = l/e$. We found that if $b < \sqrt{27}GM$, the photon will be captured by the mass.

If we align the x direction in the observer's local frame with the ϕ direction in the global frame, and the local z direction with the global r direction, then we found that $v_x^2 + v_z^2 = 1$, so that the initial angle of travel of the photon is $\sin\psi = v_x/1 = v_x$. Therefore the condition for the emitted photon to be captured is

$$\sin\psi < \frac{\sqrt{27}GM}{r}\left(1 - \frac{2GM}{r}\right)^{1/2} \tag{2}$$

and the critical angle is

$$\psi_c = \arcsin\left[\frac{\sqrt{27}GM}{r}\left(1-\frac{2GM}{r}\right)^{1/2}\right] \tag{3}$$

We need to be a bit careful in calculating the angle, since the sine is symmetric about the point $\psi = \pi/2$. To work out which angle is correct, we observe that if we emitted a photon radially outward, $\psi = 0$ and we would expect the photon to escape to infinity unless we're inside the black hole's event horizon. However, if $\psi = \pi$, the photon is emitted directly at the black hole and will always be absorbed. As we gradually decrease ψ from π, we'd expect the photon to be absorbed until we hit the critical angle, so the range of ψ that allows the photon to be absorbed starts at π and *decreases* until we hit $\psi = \psi_c$.

For example, suppose we require

$$\frac{\sqrt{27}GM}{r}\left(1-\frac{2GM}{r}\right)^{1/2} = \frac{\sqrt{2}}{2} \tag{4}$$

Then $\psi_c = \frac{\pi}{4}, \frac{3\pi}{4}$. Solving for r, we get

$$r = 6GM,\ 3\left(\sqrt{3}-1\right)GM \tag{5}$$

Since $6 > 3\left(\sqrt{3}-1\right) = 2.2$, $r = 6GM$ corresponds to $\psi_c = \frac{3\pi}{4}$ and $r = 3\left(\sqrt{3}-1\right)GM$ to $\psi_c = \frac{\pi}{4}$. That is, if we're at $r = 6GM$, then all photons emitted at an angle $\frac{3\pi}{4} < \psi < \pi$ will be absorbed.

Because the Schwarzschild metric is invariant under time-reversal (replacing dt by $-dt$ doesn't change it), we can invert the argument to show that any photon that arrives at the observer must do so from an angle less than $\frac{3\pi}{4}$, which means that the black hole will subtend an angle of $\pi/2$ to the observer at $r = 6GM$. To an observer at $r = 3\left(\sqrt{3}-1\right)GM$, the black hole subtends $\frac{3\pi}{2}$, so appears to almost wrap around the observer, with only a $\pi/2$ circle of the outside universe visible.

We can work out the values for other values of r. I'll give the angles in degrees, since most people find them easier to visualize than radians.

- $r = 2.5GM$. $\psi_c = 68.4^\circ$. The black hole occupies more than half the sky, subtending an angle of 223.3°.
- $r = 3GM$; $\psi_c = 90^\circ$, so at this distance, the black hole appears to fill exactly half the sky with an angle of 180°.
- $r = 4GM$; $\psi_c = 113.3^\circ$. We're now in the regime where the black hole occupies less than half the sky, with an angle of 133.4°.
- $r = 5GM$; $\psi_c = 126.4^\circ$, black hole subtends 107.2°.

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