

PHOTON PATH IN FLAT SPACE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 12; Problem 12.6.

The equations of motion for a photon in the Schwarzschild metric are:

$$\frac{d\phi}{dt} = \frac{1}{r^2} \left(1 - \frac{2GM}{r}\right) b \quad (1)$$

$$\left[\left(1 - \frac{2GM}{r}\right)^{-1} \frac{dr}{dt} \right]^2 + \left(1 - \frac{2GM}{r}\right) \frac{b^2}{r^2} = 1 \quad (2)$$

In flat space, $M = 0$ so these equations reduce to

$$\frac{d\phi}{dt} = \frac{b}{r^2} \quad (3)$$

$$\frac{dr}{dt} = \pm \sqrt{1 - \frac{b^2}{r^2}} \quad (4)$$

Dividing the first by the second, we get

$$\frac{d\phi}{dr} = \pm \frac{b}{r^2 \sqrt{1 - \frac{b^2}{r^2}}} \quad (5)$$

We can integrate this directly to get

$$\phi(r) = \mp \arctan \frac{b}{\sqrt{r^2 - b^2}} + k \quad (6)$$

where k is a constant of integration.

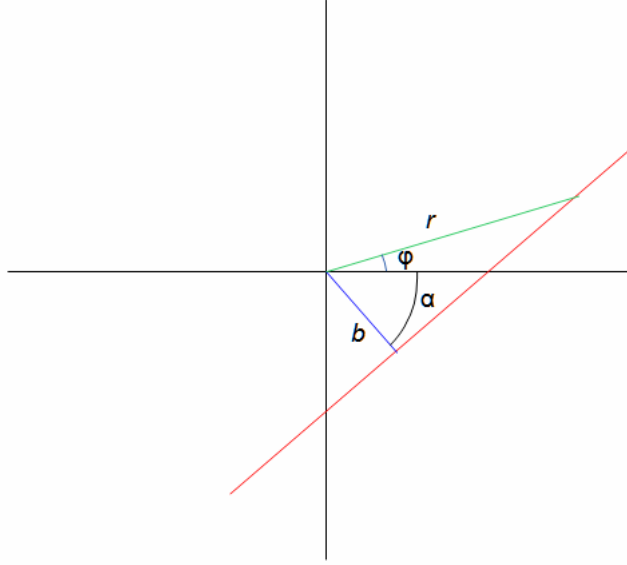
We can convert this into a more meaningful form by setting $k = \alpha + \frac{\pi}{2}$. Looking at the minus sign option first, we then get

$$\tan\left(\frac{\pi}{2} - \phi + \alpha\right) = \frac{b}{\sqrt{r^2 - b^2}} \quad (7)$$

or, using the identity $\tan\left(\frac{\pi}{2} - x\right) = 1/\tan x$

$$\tan(\phi - \alpha) = \frac{\sqrt{r^2 - b^2}}{b} \quad (8)$$

We can interpret this geometrically by looking at a plot of a straight line path in polar coordinates:



In this diagram, the line is in red, and the segment b (in blue) is the closest approach to the origin (so b is perpendicular to the line). A general point on the line is indicated by the green line r . The red side of the triangle has length $\sqrt{r^2 - b^2}$ so if we label the angles as shown, we have (taking $\alpha < 0$ since it's below the x axis) the same equation for the tangent as above. That is, the relation for the tangent describes a straight line in polar coordinates, where b is the perpendicular distance from the origin to the line, and α is the angle from the x axis to the segment b .

Taking the plus sign in the above derivation we get

$$\tan\left(-\frac{\pi}{2} + \phi - \alpha\right) = \frac{b}{\sqrt{r^2 - b^2}} \quad (9)$$

$$= -\tan\left(\frac{\pi}{2} - \phi + \alpha\right) \quad (10)$$

$$\tan(\phi - \alpha) = -\frac{\sqrt{r^2 - b^2}}{b} \quad (11)$$

This would apply if $\phi - \alpha < 0$. Thus the path of a photon in flat space is a straight line.

Incidentally, to get the same form as given by Moore in his question, we can use the trig identity $1 + \tan^2 x = \sec^2 x$:

$$1 + \tan^2(\phi - \alpha) = \frac{r^2}{b^2} \quad (12)$$

$$\cos(\phi - \alpha) = \frac{b}{r} \quad (13)$$

$$\phi = \arccos \frac{b}{r} + \alpha \quad (14)$$