

## PHOTON PATH IN FLAT SPACE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 12; Problem 12.6.

The equations of motion for a photon in the Schwarzschild metric are:

$$(0.1) \quad \frac{d\phi}{dt} = \frac{1}{r^2} \left(1 - \frac{2GM}{r}\right) b$$

$$(0.2) \quad \left[ \left(1 - \frac{2GM}{r}\right)^{-1} \frac{dr}{dt} \right]^2 + \left(1 - \frac{2GM}{r}\right) \frac{b^2}{r^2} = 1$$

In flat space,  $M = 0$  so these equations reduce to

$$(0.3) \quad \frac{d\phi}{dt} = \frac{b}{r^2}$$

$$(0.4) \quad \frac{dr}{dt} = \pm \sqrt{1 - \frac{b^2}{r^2}}$$

Dividing the first by the second, we get

$$(0.5) \quad \frac{d\phi}{dr} = \pm \frac{b}{r^2 \sqrt{1 - \frac{b^2}{r^2}}}$$

We can integrate this directly to get

$$(0.6) \quad \phi(r) = \mp \arctan \frac{b}{\sqrt{r^2 - b^2}} + k$$

where  $k$  is a constant of integration.

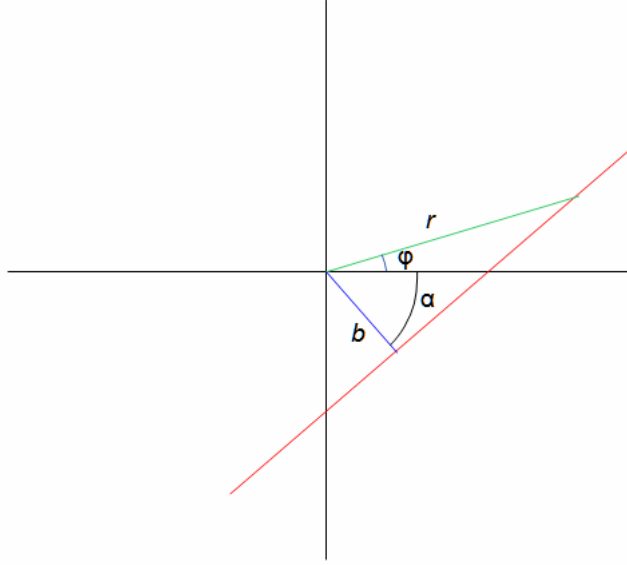
We can convert this into a more meaningful form by setting  $k = \alpha + \frac{\pi}{2}$ . Looking at the minus sign option first, we then get

$$(0.7) \quad \tan \left( \frac{\pi}{2} - \phi + \alpha \right) = \frac{b}{\sqrt{r^2 - b^2}}$$

or, using the identity  $\tan \left( \frac{\pi}{2} - x \right) = 1 / \tan x$

$$(0.8) \quad \tan(\phi - \alpha) = \frac{\sqrt{r^2 - b^2}}{b}$$

We can interpret this geometrically by looking at a plot of a straight line path in polar coordinates:



In this diagram, the line is in red, and the segment  $b$  (in blue) is the closest approach to the origin (so  $b$  is perpendicular to the line). A general point on the line is indicated by the green line  $r$ . The red side of the triangle has length  $\sqrt{r^2 - b^2}$  so if we label the angles as shown, we have (taking  $\alpha < 0$  since it's below the  $x$  axis) the same equation for the tangent as above. That is, the relation for the tangent describes a straight line in polar coordinates, where  $b$  is the perpendicular distance from the origin to the line, and  $\alpha$  is the angle from the  $x$  axis to the segment  $b$ .

Taking the plus sign in the above derivation we get

$$(0.9) \quad \tan\left(-\frac{\pi}{2} + \phi - \alpha\right) = \frac{b}{\sqrt{r^2 - b^2}}$$

$$(0.10) \quad = -\tan\left(\frac{\pi}{2} - \phi + \alpha\right)$$

$$(0.11) \quad \tan(\phi - \alpha) = -\frac{\sqrt{r^2 - b^2}}{b}$$

This would apply if  $\phi - \alpha < 0$ . Thus the path of a photon in flat space is a straight line.

Incidentally, to get the same form as given by Moore in his question, we can use the trig identity  $1 + \tan^2 x = \sec^2 x$ :

$$(0.12) \quad 1 + \tan^2(\phi - \alpha) = \frac{r^2}{b^2}$$

$$(0.13) \quad \cos(\phi - \alpha) = \frac{b}{r}$$

$$(0.14) \quad \phi = \arccos \frac{b}{r} + \alpha$$