PHOTON PATH IN FLAT SPACE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 12; Problem 12.6.

The equations of motion for a photon in the Schwarzschild metric are:

$$\frac{d\phi}{dt}=\frac{1}{r^{2}}\left(1-\frac{2GM}{r}\right)b \tag{1}$$

$$\left[\left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{dt}\right]^{2}+\left(1-\frac{2GM}{r}\right)\frac{b^{2}}{r^{2}}=1 \tag{2}$$

In flat space, $M=0$ so these equations reduce to

$$\frac{d\phi}{dt} = \frac{b}{r^{2}} \tag{3}$$

$$\frac{dr}{dt} = \pm\sqrt{1-\frac{b^{2}}{r^{2}}} \tag{4}$$

Dividing the first by the second, we get

$$\frac{d\phi}{dr}=\pm\frac{b}{r^{2}\sqrt{1-\frac{b^{2}}{r^{2}}}} \tag{5}$$

We can integrate this directly to get

$$\phi\left(r\right)=\mp\arctan\frac{b}{\sqrt{r^{2}-b^{2}}}+k \tag{6}$$

where k is a constant of integration.

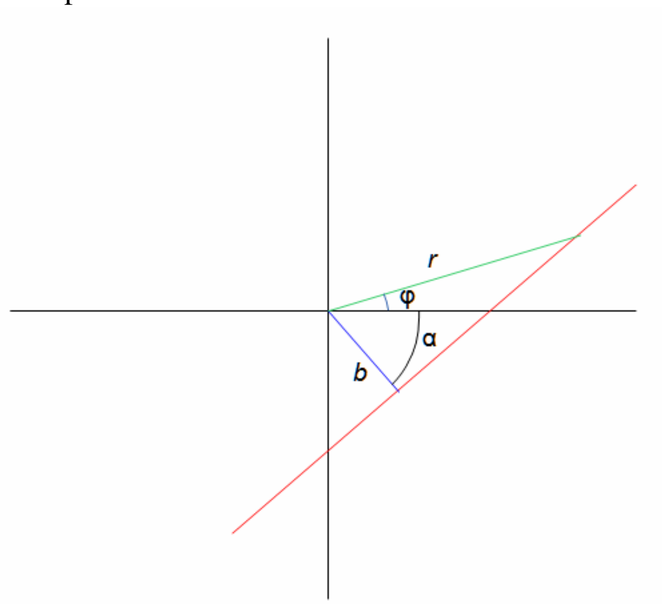
We can convert this into a more meaningful form by setting $k=\alpha+\frac{\pi}{2}$. Looking at the minus sign option first, we then get

$$\tan\left(\frac{\pi}{2}-\phi+\alpha\right)=\frac{b}{\sqrt{r^{2}-b^{2}}} \tag{7}$$

or, using the identity $\tan\left(\frac{\pi}{2}-x\right)=1/\tan x$

$$\tan(\phi-\alpha)=\frac{\sqrt{r^2-b^2}}{b} \tag{8}$$

We can interpret this geometrically by looking at a plot of a straight line path in polar coordinates:



In this diagram, the line is in red, and the segment b (in blue) is the closest approach to the origin (so b is perpendicular to the line). A general point on the line is indicated by the green line r. The red side of the triangle has length $\sqrt{r^2-b^2}$ so if we label the angles as shown, we have (taking $\alpha<0$ since it's below the x axis) the same equation for the tangent as above. That is, the relation for the tangent describes a straight line in polar coordinates, where b is the perpendicular distance from the origin to the line, and α is the angle from the x axis to the segment b.

Taking the plus sign in the above derivation we get

$$\begin{aligned}
\tan\left(-\frac{\pi}{2}+\phi-\alpha\right) &= \frac{b}{\sqrt{r^2-b^2}} && (9)\\
&= -\tan\left(\frac{\pi}{2}-\phi+\alpha\right) && (10)\\
\tan(\phi-\alpha) &= -\frac{\sqrt{r^2-b^2}}{b} && (11)
\end{aligned}$$

This would apply if $\phi-\alpha<0$. Thus the path of a photon in flat space is a straight line.

Incidentally, to get the same form as given by Moore in his question, we can use the trig identity $1+\tan^2 x = \sec^2 x$:

$$1+\tan^2(\phi-\alpha) = \frac{r^2}{b^2} \tag{12}$$

$$\cos(\phi-\alpha) = \frac{b}{r} \tag{13}$$

$$\phi = \arccos\frac{b}{r}+\alpha \tag{14}$$