APPARENT SIZE OF A BLACK HOLE TO A MOVING OBSERVER

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 12; Problem 12.7.

We've seen what the view of black hole looks like for a stationary observer at various distances from the black hole. We can do similar calculations for an observer falling in radially from infinity. For such an observer, we've already worked out the four-velocity components:

$$\frac{dt}{d\tau} = e\left(1-\frac{2GM}{r}\right)^{-1} \tag{1}$$

$$\frac{d\theta}{d\tau} = 0 \tag{2}$$

$$\frac{d\phi}{d\tau} = \frac{l}{r^2\sin^2\theta}=\frac{l}{r^2} \tag{3}$$

$$\frac{dr}{d\tau} = \pm\sqrt{\frac{2GM}{r}} \tag{4}$$

For a particle starting at rest at infinity and moving radially inwards, $dr/d\tau<0$, $e=1$ and $l=0$, so these equations reduce to

$$\frac{dt}{d\tau} = \left(1-\frac{2GM}{r}\right)^{-1} \tag{5}$$

$$\frac{d\theta}{d\tau} = 0 \tag{6}$$

$$\frac{d\phi}{d\tau} = 0 \tag{7}$$

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r}} \tag{8}$$

These are the components of the basis vector $\mathbf{o}_t$ in the Schwarzschild frame:

$$\mathbf{o}_t=\left[\left(1-\frac{2GM}{r}\right)^{-1},-\sqrt{\frac{2GM}{r}},0,0\right] \tag{9}$$

Note that this vector is already normalized, since

$$\mathbf{o}_t \cdot \mathbf{o}_t = -\left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-2} + \left(1-\frac{2GM}{r}\right)^{-1}\frac{2GM}{r} \tag{10}$$

$$= -1 \tag{11}$$

From this, we can work out the three spatial basis vectors. For $\mathbf{o}_z$, we know that $\mathbf{o}_t \cdot \mathbf{o}_z = 0$ and, since the z axis is aligned (by definition) with the r direction, $o_z^\phi = o_z^\theta = 0$, so we get

$$\mathbf{o}_t \cdot \mathbf{o}_z = g_{ij} o_t^i o_z^j \tag{12}$$

$$= -\left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-1} o_z^t - \left(1-\frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}}\, o_z^r = 0 \tag{13}$$

$$o_z^t = -\left(1-\frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}}\, o_z^r \tag{14}$$

We also have the normalization condition $\mathbf{o}_z \cdot \mathbf{o}_z = 1$ which gives us

$$\mathbf{o}_z \cdot \mathbf{o}_z = \left[-\left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-2}\frac{2GM}{r} + \left(1-\frac{2GM}{r}\right)^{-1}\right](o_z^r)^2 \tag{15}$$

$$= (o_z^r)^2 = 1 \tag{16}$$

$$o_z^r = 1 \tag{17}$$

We choose $+1$ for o_z^r since the z axis is aligned with the $+r$ direction. Thus:

$$\mathbf{o}_z = \left[-\left(1-\frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}}, 1, 0, 0\right] \tag{18}$$

Since $o_t^\phi = o_t^\theta = 0$, the condition $\mathbf{o}_x \cdot \mathbf{o}_t = \mathbf{o}_y \cdot \mathbf{o}_t = 0$ tells us that $o_x^t = o_y^t = 0$, so the normalization condition then says that $\mathbf{o}_x$ and $\mathbf{o}_y$ are the same as before, namely

$$\mathbf{o}_x = \left[0,0,0,\frac{1}{r\sin\theta}\right] = \left[0,0,0,\frac{1}{r}\right] \tag{19}$$

$$\mathbf{o}_y = \left[0,0,-\frac{1}{r},0\right] \tag{20}$$

We can now follow the same procedure as before to calculate the critical angle at which a photon emitted by the observer is absorbed by the black hole. We have the photon's four-momentum:

$$\begin{aligned}
p^t &= E\left(1-\frac{2GM}{r}\right)^{-1}\frac{dt}{dt} = E\left(1-\frac{2GM}{r}\right)^{-1} && (21)\\
p^r &= E\left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{dt} && (22)\\
&= \pm E\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}} && (23)\\
p^\theta &= 0 && (24)\\
p^\phi &= E\left(1-\frac{2GM}{r}\right)^{-1}\frac{d\phi}{dt} && (25)\\
&= E\frac{b}{r^2} && (26)
\end{aligned}$$

The 3-velocity components as measured by the observer are

$$\begin{aligned}
v^i &= \frac{p^i}{p^t} && (27)\\
&= \frac{\mathbf{o}_i\cdot\mathbf{p}}{-\mathbf{o}_t\cdot\mathbf{p}} && (28)
\end{aligned}$$

Using our basis vectors from above, we get

$$-\mathbf{o}_t\cdot\mathbf{p} = E\left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-2} \pm E\left(1-\frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}}\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}} \quad (29)$$

$$= E\left(1-\frac{2GM}{r}\right)^{-1}\left[1\pm\sqrt{\frac{2GM}{r}}\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}}\right] \quad (30)$$

$$\mathbf{o}_x\cdot\mathbf{p} = E\frac{b}{r} \quad (31)$$

The sine of the emitted angle is, as before

$$\sin\psi = \frac{v_x}{1} = \frac{\mathbf{o}_x \cdot \mathbf{p}}{-\mathbf{o}_t \cdot \mathbf{p}} \tag{32}$$

$$= \frac{b}{r}\left(1 - \frac{2GM}{r}\right)\left[1 \pm \sqrt{\frac{2GM}{r}}\sqrt{1 - \left(1 - \frac{2GM}{r}\right)\frac{b^2}{r^2}}\right]^{-1} \tag{33}$$

As a check, we can also calculate the cosine:

$$\cos\psi = v_z = \frac{\mathbf{o}_z \cdot \mathbf{p}}{-\mathbf{o}_t \cdot \mathbf{p}} \tag{34}$$

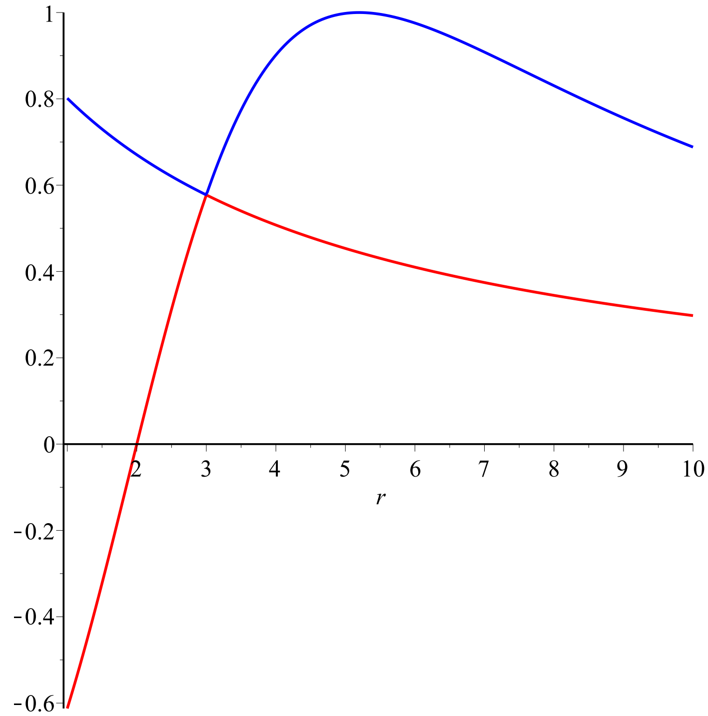
$$= \left[\sqrt{\frac{2GM}{r}} \pm \sqrt{1 - \left(1 - \frac{2GM}{r}\right)\frac{b^2}{r^2}}\right]\left[1 \pm \sqrt{\frac{2GM}{r}}\sqrt{1 - \left(1 - \frac{2GM}{r}\right)\frac{b^2}{r^2}}\right]^{-1} \tag{35}$$

After a bit of algebra, it can be confirmed that $\sin^2\psi + \cos^2\psi = 1$ which is reassuring.

The critical angle occurs when the impact parameter $b = \sqrt{27}GM$, so

$$\sin\psi_c = \frac{\sqrt{27}GM}{r}\left(1 - \frac{2GM}{r}\right)\left[1 \pm \sqrt{\frac{2GM}{r}}\sqrt{1 - \left(1 - \frac{2GM}{r}\right)\frac{27G^2M^2}{r^2}}\right]^{-1} \tag{36}$$

Unlike the case for a stationary observer which is defined only for $r > 2GM$, this formula is actually well-defined for all $r > 0$, although $\sin\psi_c < 0$ for $r < 2GM$. We'll plot $\sin\psi_c$ versus r (in units of GM) for both signs, with the plus sign in red and the minus sign in blue. We see that something odd happens at $r = 3GM$:

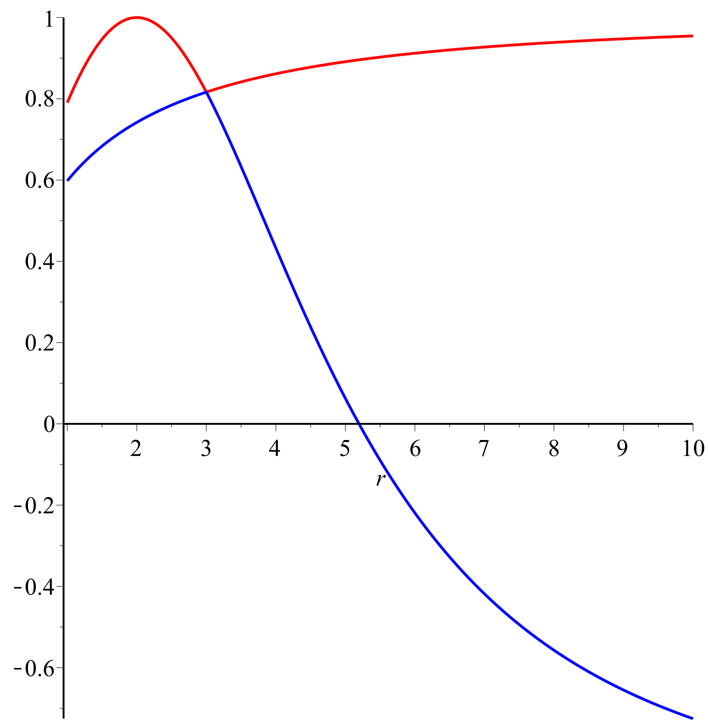


I'm not totally sure of the interpretation, but if we look at the analysis of the stationary observer that we did earlier, we see that at $r = 3GM$, the critical emission angle is $\psi_c = 90°$. That is, for $r > 3GM$, the photons at the critical angle are emitted such $dr/dt < 0$, while for $r < 3GM$, they are emitted with $dr/dt > 0$. This means that we should take the minus sign for p_r in the former case, and the plus sign in the latter. Since the observer is stationary, it is in the same frame as the global Schwarzschild frame.

When dealing with a moving observer, we are still doing the calculations in the global frame (that is, the frame of the central mass M); it is only the local basis vectors that have changed due to the motion of the observer. Therefore, the switch from positive to negative p_r still happens at $r = 3GM$, since we are using $\mathbf{p}$ as calculated in the global frame. In the plot, therefore, we should use the red curve for $r < 3GM$ and the blue curve for $r > 3GM$.

To the *moving* observer, however, the angle subtended by the black hole is different from that for a stationary observer at the same distance from the black hole. For example, the maximum of $\sin\psi_c = 1$ occurs at $r = 5.196GM$ so it is at that radius that the critical angle *relative to the moving observer* is 90°.

The plot of $\cos\psi_c$ allows us to determine the quadrant of ψ_c:



At other radii, we have:

- $r = 4GM$; $\psi_c = 64.355°$
- $r = 3GM$; $\psi_c = 35.264°$
- $r = 2GM$; $\psi_c = 0°$
- $r = GM$; $\psi_c = -37.771°$

I'm not sure what to make of the last result, since I'd imagine $r = GM$ is inside the black hole, where presumably the Schwarzschild metric breaks down.

From the formula $E_{obs} = -\mathbf{o}_t \cdot \mathbf{p}$ we can work out the observed energy of a photon fired radially from infinity at the observer. If the photon is coming in on a radial line, the impact parameter $b = 0$ and $p_r = -E_\infty$ so we get

$$E_{obs} = E_\infty \left(1 - \sqrt{\frac{2GM}{r}}\right)\left(1 - \frac{2GM}{r}\right)^{-1} \tag{37}$$

$$= \frac{E_\infty}{1 + \sqrt{\frac{2GM}{r}}} \tag{38}$$

where E_∞ is the photon's energy as observed at infinity. Since $E_{obs} < E_\infty$ for all r, the light is always red-shifted. The fractional change in wavelength is

$$\frac{\Delta\lambda}{\lambda_{\infty}} = \left(\frac{1}{E_{\infty}}-\frac{1}{E_{obs}}\right)E_{\infty} \tag{39}$$

$$= -\sqrt{\frac{2GM}{r}} \tag{40}$$

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