CIRCULAR ORBIT: APPEARANCE TO A FALLING OBSERVER

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 12; Problem 12.8.

Suppose we have an observer falling in towards a black hole, having started at rest at infinity. We've seen that the local flat frame of such an observer has basis vectors whose components in the global frame are given by

$$\mathbf{o}_t = \left[\left(1-\frac{2GM}{r}\right)^{-1},-\sqrt{\frac{2GM}{r}},0,0\right] \tag{1}$$

$$\mathbf{o}_z = \left[-\left(1-\frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}},1,0,0\right] \tag{2}$$

$$\mathbf{o}_x = \left[0,0,0,\frac{1}{r}\right] \tag{3}$$

$$\mathbf{o}_y = \left[0,0,-\frac{1}{r},0\right] \tag{4}$$

Now suppose that as this observer crosses the point $r=6GM$, an object in a circular orbit at that radius flies past him. What is the (ordinary) velocity of this object as seen by the observer?

To answer this, we need the four-momentum of the object in the circular orbit. We can get this from the conserved quantities e and l.

$$r^2\frac{d\phi}{d\tau} = l \tag{5}$$

$$\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau} = e \tag{6}$$

For a circular orbit, we can get l and e using our earlier formulas:

$$
\begin{aligned}
l^2 &= \frac{r^2GM}{r-3GM} && (7)\\
e &= \left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{1/2} && (8)
\end{aligned}
$$

For $r = 6GM$, we get

$$
\begin{aligned}
l &= \sqrt{12}GM && (9)\\
e &= \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} && (10)
\end{aligned}
$$

Plugging these into the earlier equations, we get

$$
\begin{aligned}
\frac{d\phi}{d\tau} &= \frac{\sqrt{3}}{18GM} && (11)\\
\frac{dt}{d\tau} &= \sqrt{2} && (12)\\
\frac{dr}{d\tau} &= 0 && (13)\\
\frac{d\theta}{d\tau} &= 0 && (14)
\end{aligned}
$$

where the last two follow from the fact that the orbit is circular (so r is constant) and in the equatorial plane. Thus the four-momentum (in the global frame) is

$$\mathbf{p} = m\left[\sqrt{2}, 0, 0, \frac{\sqrt{3}}{18GM}\right] \tag{15}$$

where m is the mass of the orbiting object.

We can now get the components of the velocity relative to the observer from

$$v_i = \frac{\mathbf{o}_i \cdot \mathbf{p}}{-\mathbf{o}_t \cdot \mathbf{p}} \tag{16}$$

We therefore have (leaving out all terms that are zero):

$$
\begin{aligned}
v_x &= \frac{g_{\phi\phi} o_x^{\phi} p^{\phi}}{-g_{tt} o_t^t p^t} && (17)\\
&= -\frac{r^2 \frac{1}{r} \frac{\sqrt{3}m}{18GM}}{-\left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-1}\sqrt{2}m} && (18)\\
&= \frac{1}{\sqrt{6}} && (19)\\
v_y &= 0 && (20)\\
v_z &= \frac{g_{tt} o_z^t p^t}{-g_{tt} o_t^t p^t} && (21)\\
&= \frac{1}{\sqrt{3}} && (22)
\end{aligned}
$$

The speed is then

$$
v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \frac{1}{\sqrt{2}} \tag{23}
$$