LOCAL FLAT FRAME FOR A CIRCULAR ORBIT

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 12; Problem 12.9.

Here's another example of calculating a flat local reference frame in a global Schwarzschild metric. We'll follow the same procedure as in the case of the observer falling radially inwards. This time, we're interested in an observer in a circular orbit of radius r. First, we need to work out the four-velocity in Schwarzschild coordinates. We can get this from the conserved quantities e and l.

$$r^2\frac{d\phi}{d\tau} = l \tag{1}$$

$$\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau} = e \tag{2}$$

For a circular orbit, we can get l and e using our earlier formulas:

$$l^2 = \frac{r^2GM}{r-3GM} \tag{3}$$

$$e = \left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{1/2} \tag{4}$$

Therefore

$$\frac{d\phi}{d\tau} = \frac{l}{r^2} \tag{5}$$

$$= \frac{1}{r}\sqrt{\frac{GM}{r-3GM}} \tag{6}$$

$$\frac{dt}{d\tau} = \frac{e}{1-2GM/r} \tag{7}$$

$$= \left(1-\frac{2GM}{r}\right)^{-1}\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{1/2} \tag{8}$$

Since there is no motion in the radial or θ directions, the four-velocity is

(9)
$$\mathbf{o}_t=\mathbf{u}=\left[\left(\left(1-\frac{2GM}{r}\right)^{-1}\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]\right)^{1/2},0,0,\frac{1}{r}\sqrt{\frac{GM}{r-3GM}}\right]$$

As a check, we can verify that $\mathbf{u}\cdot\mathbf{u}=-1$ (this is most easily done with Maple, but if you're persistent it can be done by hand).

Now we align the x, y and z local directions with the global ϕ, $-\theta$ and r directions respectively. For $\mathbf{o}_y$ and $\mathbf{o}_z$, this means they will have the same global components as for a stationary observer, so that

$$\mathbf{o}_y = \left[0,0,-\frac{1}{r},0\right] \tag{10}$$

$$\mathbf{o}_z = \left[0,\sqrt{1-\frac{2GM}{r}},0,0\right] \tag{11}$$

To find $\mathbf{o}_x$, we start with $\mathbf{o}_x\cdot\mathbf{o}_t=0$, which gives us

$$g_{tt}o_t^t o_x^t+g_{\phi\phi}o_t^\phi o_x^\phi=0 \tag{12}$$

(13)
$$-\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{1/2}o_x^t+r\sqrt{\frac{GM}{r-3GM}}o_x^\phi=0$$

$$\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{-1/2}r\sqrt{\frac{GM}{r-3GM}}o_x^\phi=o_x^t \tag{14}$$

Now we can use normalization, so that $\mathbf{o}_x\cdot\mathbf{o}_x=1$. We get

(15)
$$\left\{1-\left(1-\frac{2GM}{r}\right)\frac{GM}{r-3GM}\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{-1}\right\}r^2\left(o_x^\phi\right)^2=1$$

The algebra is tedious so is best handled with Maple, and after simplifying, we get

$$o_x^\phi=\frac{1}{r}\sqrt{\frac{r-2GM}{r-3GM}} \tag{16}$$

We can substitute this back into the earlier equation to get o_x^t, and, after simplifying, we get

$$\mathbf{o}_x = \left[\sqrt{\frac{rGM}{(r-2GM)(r-3GM)}}, 0, 0, \frac{1}{r}\sqrt{\frac{r-2GM}{r-3GM}}\right] \tag{17}$$

The formula breaks down for $r \leq 3GM$, which is presumably because if a photon approaches more closely than that, it spirals into $r = 0$. In particular, for $r \leq 3GM$, it is not possible for dr/dt to be zero, which we assumed in this derivation.