

LOCAL FLAT FRAME FOR A CIRCULAR ORBIT

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 12; Problem 12.9.

Here's another example of calculating a flat local reference frame in a global Schwarzschild metric. We'll follow the same procedure as in the case of the observer falling radially inwards. This time, we're interested in an observer in a circular orbit of radius r . First, we need to work out the four-velocity in Schwarzschild coordinates. We can get this from the conserved quantities e and l .

$$(0.1) \quad r^2 \frac{d\phi}{d\tau} = l$$

$$(0.2) \quad \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau} = e$$

For a circular orbit, we can get l and e using our earlier formulas:

$$(0.3) \quad l^2 = \frac{r^2 GM}{r - 3GM}$$

$$(0.4) \quad e = \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right)\right]^{1/2}$$

Therefore

$$(0.5) \quad \frac{d\phi}{d\tau} = \frac{l}{r^2}$$

$$(0.6) \quad = \frac{1}{r} \sqrt{\frac{GM}{r - 3GM}}$$

$$(0.7) \quad \frac{dt}{d\tau} = \frac{e}{1 - 2GM/r}$$

$$(0.8) \quad = \left(1 - \frac{2GM}{r}\right)^{-1} \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right)\right]^{1/2}$$

Since there is no motion in the radial or θ directions, the four-velocity is

$$(0.9) \quad \mathbf{o}_t = \mathbf{u} = \left[\left(1 - \frac{2GM}{r}\right)^{-1} \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right) \right]^{1/2}, 0, 0, \frac{1}{r} \sqrt{\frac{GM}{r-3GM}} \right]$$

As a check, we can verify that $\mathbf{u} \cdot \mathbf{u} = -1$ (this is most easily done with Maple, but if you're persistent it can be done by hand).

Now we align the x , y and z local directions with the global ϕ , $-\theta$ and r directions respectively. For \mathbf{o}_y and \mathbf{o}_z , this means they will have the same global components as for a stationary observer, so that

$$(0.10) \quad \mathbf{o}_y = \left[0, 0, -\frac{1}{r}, 0 \right]$$

$$(0.11) \quad \mathbf{o}_z = \left[0, \sqrt{1 - \frac{2GM}{r}}, 0, 0 \right]$$

To find \mathbf{o}_x , we start with $\mathbf{o}_x \cdot \mathbf{o}_t = 0$, which gives us

$$(0.12) \quad g_{tt} o_t^t o_x^t + g_{\phi\phi} o_t^\phi o_x^\phi = 0$$

$$(0.13)$$

$$- \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right) \right]^{1/2} o_x^t + r \sqrt{\frac{GM}{r-3GM}} o_x^\phi = 0$$

$$(0.14) \quad \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right) \right]^{-1/2} r \sqrt{\frac{GM}{r-3GM}} o_x^\phi = o_x^t$$

Now we can use normalization, so that $\mathbf{o}_x \cdot \mathbf{o}_x = 1$. We get

$$(0.15)$$

$$\left\{ 1 - \left(1 - \frac{2GM}{r}\right) \frac{GM}{r-3GM} \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right) \right]^{-1} \right\} r^2 (o_x^\phi)^2 = 1$$

The algebra is tedious so is best handled with Maple, and after simplifying, we get

$$(0.16) \quad o_x^\phi = \frac{1}{r} \sqrt{\frac{r-2GM}{r-3GM}}$$

We can substitute this back into the earlier equation to get o_x^t , and, after simplifying, we get

$$(0.17) \quad \mathbf{o}_x = \left[\sqrt{\frac{rGM}{(r-2GM)(r-3GM)}}, 0, 0, \frac{1}{r} \sqrt{\frac{r-2GM}{r-3GM}} \right]$$

The formula breaks down for $r \leq 3GM$, which is presumably because if a photon approaches more closely than that, it spirals into $r = 0$. In particular, for $r \leq 3GM$, it is not possible for dr/dt to be zero, which we assumed in this derivation.