LOCAL FLAT FRAME FOR A CIRCULAR ORBIT

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 12; Problem 12.9.

Here's another example of calculating a flat local reference frame in a global Schwarzschild metric. We'll follow the same procedure as in the case of the observer falling radially inwards. This time, we're interested in an observer in a circular orbit of radius r. First, we need to work out the four-velocity in Schwarzschild coordinates. We can get this from the conserved quantities e and l.

$$r^2\frac{d\phi}{d\tau} = l \tag{1}$$

$$\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau} = e \tag{2}$$

For a circular orbit, we can get l and e using our earlier formulas:

$$l^2 = \frac{r^2 GM}{r-3GM} \tag{3}$$

$$e = \left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{1/2} \tag{4}$$

Therefore

$$\frac{d\phi}{d\tau} = \frac{l}{r^2} \tag{5}$$

$$= \frac{1}{r}\sqrt{\frac{GM}{r-3GM}} \tag{6}$$

$$\frac{dt}{d\tau} = \frac{e}{1-2GM/r} \tag{7}$$

$$= \left(1-\frac{2GM}{r}\right)^{-1}\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{1/2} \tag{8}$$

Since there is no motion in the radial or θ directions, the four-velocity is

$$\mathbf{o}_t = \mathbf{u} = \left[\left(\left(1-\frac{2GM}{r}\right)^{-1}\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]\right)^{1/2}, 0, 0, \frac{1}{r}\sqrt{\frac{GM}{r-3GM}}\right] \tag{9}$$

This can be simplified a bit by using some algebra. The first component is

$$\left(1-\frac{2GM}{r}\right)^{-1}\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{1/2} = \left[\frac{r-3GM-GM+4(GM)^2/r}{(r-3GM)(1-2GM/r)^2}\right]^{1/2} \tag{10}$$

$$= \left[\frac{r\left(1-4GM/r+4(GM/r)^2\right)}{(r-3GM)(1-2GM/r)^2}\right]^{1/2} \tag{11}$$

$$= \left[\frac{r(1-2GM/r)^2}{(r-3GM)(1-2GM/r)^2}\right]^{1/2} \tag{12}$$

$$= \frac{1}{\sqrt{1-\frac{3GM}{r}}} \tag{13}$$

We can therefore rewrite 9 as

$$\mathbf{o}_t = \mathbf{u} = \frac{1}{\sqrt{1-\frac{3GM}{r}}}\left[1, 0, 0, \sqrt{\frac{GM}{r^3}}\right] \tag{14}$$

As a check, we can verify that $\mathbf{u}\cdot\mathbf{u} = -1$ (this is most easily done with Maple, but if you're persistent it can be done by hand).

Now we align the x, y and z local directions with the global ϕ, $-\theta$ and r directions respectively. For $\mathbf{o}_y$ and $\mathbf{o}_z$, this means they will have the same global components as for a stationary observer, so that

$$\mathbf{o}_y = \left[0, 0, -\frac{1}{r}, 0\right] \tag{15}$$

$$\mathbf{o}_z = \left[0, \sqrt{1-\frac{2GM}{r}}, 0, 0\right] \tag{16}$$

To find $\mathbf{o}_x$, we start with $\mathbf{o}_x \cdot \mathbf{o}_t = 0$, which gives us

$$g_{tt}o_t^t o_x^t + g_{\phi\phi}o_t^\phi o_x^\phi = 0 \tag{17}$$

$$-\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{1/2} o_x^t + r\sqrt{\frac{GM}{r-3GM}}o_x^\phi = 0 \tag{18}$$

$$\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{-1/2} r\sqrt{\frac{GM}{r-3GM}}o_x^\phi = o_x^t \tag{19}$$

Now we can use normalization, so that $\mathbf{o}_x \cdot \mathbf{o}_x = 1$. We get

$$\left\{1-\left(1-\frac{2GM}{r}\right)\frac{GM}{r-3GM}\left[1-\frac{GM}{r}\left(1-\frac{3GM}{r}\right)^{-1}\left(1-\frac{4GM}{r}\right)\right]^{-1}\right\} r^2\left(o_x^\phi\right)^2 = 1 \tag{20}$$

The algebra is tedious so is best handled with Maple, and after simplifying, we get

$$o_x^\phi = \frac{1}{r}\sqrt{\frac{r-2GM}{r-3GM}} \tag{21}$$

We can substitute this back into the earlier equation to get o_x^t, and, after simplifying, we get

$$\mathbf{o}_x = \left[\sqrt{\frac{rGM}{(r-2GM)(r-3GM)}}, 0, 0, \frac{1}{r}\sqrt{\frac{r-2GM}{r-3GM}}\right] \tag{22}$$

The formula breaks down for $r \leq 3GM$, which is presumably because if a photon approaches more closely than that, it spirals into $r=0$. In particular, for $r \leq 3GM$, it is not possible for dr/dt to be zero, which we assumed in this derivation.