

DEFLECTION OF LIGHT BY A MASS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 13; Problem 13.1.

We've seen how to derive the equations of motion for a photon in a gravitational field, so we can now apply these equations to study the deflection of light as it passes a massive object. The equations of motion are

$$(0.1) \quad \frac{d\phi}{dt} = \frac{1}{r^2} \left(1 - \frac{2GM}{r}\right) b$$

$$(0.2) \quad \left[\frac{1}{b} \left(1 - \frac{2GM}{r}\right)^{-1} \frac{dr}{dt} \right]^2 + \frac{1}{r^2} \left(1 - \frac{2GM}{r}\right) = \frac{1}{b^2}$$

These equations describe a photon's motion for $r > 2GM$, but when we're discussing the deflection of light as it passes a star such as the sun, typically $r \gg 2GM$. For example, for the Sun, $2GM = 2.954 \text{ km}$ and the radius of the sun is $6.955 \times 10^5 \text{ km}$ so a photon grazing the surface of the Sun as it passed has a value of $r/2GM = 4.25 \times 10^6$. In this limiting case, we can make a few approximations to get an idea of how much light is deflected as it passes a massive object.

Things are a bit easier if we work with the variable $u \equiv 1/r$, so we can use the chain rule to write

$$(0.3) \quad \frac{dr}{dt} = \frac{dr}{du} \frac{du}{d\phi} \frac{d\phi}{dt}$$

$$(0.4) \quad = -\frac{1}{u^2} \frac{du}{d\phi} u^2 (1 - 2GMu) b$$

$$(0.5) \quad = -(1 - 2GMu) b \frac{du}{d\phi}$$

Plugging this into the radial equation of motion above, we get

$$(0.6) \quad \left(\frac{du}{d\phi}\right)^2 + u^2(1 - 2GMu) = \frac{1}{b^2}$$

$$(0.7) \quad \left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{1}{b^2} + 2GMu^3$$

We can take the derivative of this with respect to ϕ and get

$$(0.8) \quad 2\frac{du}{d\phi}\frac{d^2u}{d\phi^2} + 2u\frac{du}{d\phi} = 6GMu^2\frac{du}{d\phi}$$

$$(0.9) \quad \frac{d^2u}{d\phi^2} + u = 3GMu^2$$

If $M = 0$ (that is, space is flat), this equation reduces to

$$(0.10) \quad \frac{d^2u}{d\phi^2} + u = 0$$

which has the general solution

$$(0.11) \quad u = A \sin \phi + B \cos \phi$$

If we orient the coordinate system so that $\phi = \frac{\pi}{2}$ is the angle of closest approach to the origin, then $u = 1/r$ is a maximum at that angle so

$$(0.12) \quad A \cos \frac{\pi}{2} - B \sin \frac{\pi}{2} = 0$$

$$(0.13) \quad B = 0$$

and we can set $A \equiv 1/r_c$ where r_c is the distance of closest approach to the origin:

$$(0.14) \quad u = \frac{1}{r} = \frac{1}{r_c} \sin \phi$$

This is, in fact, the equation of a straight line in polar coordinates as can be verified by drawing the right-angled triangle with sides r (as the hypotenuse) and $r_c = r \sin \phi$ (the third side is just $r \cos \phi$). From the equation above, with $M = 0$, we have

$$(0.15) \quad \left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{1}{r_c^2} (\cos^2 \phi + \sin^2 \phi) = \frac{1}{r_c^2} = \frac{1}{b^2}$$

Thus $r_c = b$, which is the impact parameter, and gives the distance of closest approach.

Now let's put the mass back in and apply the assumption $r_c \gg 2GM$. In this case, we expect that u will be fairly close to the flat space solution, so we can try a solution of the form

$$(0.16) \quad u = \frac{1}{r_c} \sin \phi + \frac{w(\phi)}{r_c}$$

where w is some function that should be small for all values of ϕ . Plugging this into the equation of motion above:

$$(0.17) \quad \frac{d^2 u}{d\phi^2} + u = 3GMu^2$$

(0.18)

$$-\frac{1}{r_c} \sin \phi + \frac{1}{r_c} \frac{d^2 w}{d\phi^2} + \frac{1}{r_c} \sin \phi + \frac{w(\phi)}{r_c} = \frac{3GM}{r_c^2} (\sin^2 \phi + 2w \sin \phi + w^2)$$

Because of our assumption $r_c \gg 2GM$, $r_c \gg 3GM$ as well, so if we're saving only up to first-order terms in the 'small' quantities, only the first term on the RHS will contribute. We then get

$$(0.19) \quad \frac{d^2 w}{d\phi^2} + w \approx \frac{3GM}{r_c} \sin^2 \phi$$

$$(0.20) \quad = \frac{3GM}{2r_c} (1 - \cos 2\phi)$$

This differential equation can be solved by assuming a solution of form $w = A + B \cos 2\phi$:

$$(0.21) \quad -4B \cos 2\phi + A + B \cos 2\phi = \frac{3GM}{2r_c} (1 - \cos 2\phi)$$

from which we get

$$(0.22) \quad A = \frac{3GM}{2r_c}$$

$$(0.23) \quad B = \frac{GM}{2r_c}$$

So

$$(0.24) \quad u = \frac{1}{r_c} \left(\sin \phi + \frac{3GM}{2r_c} \left(1 + \frac{1}{3} \cos 2\phi \right) \right)$$

We saw above that in flat space, $r_c = b$, the impact parameter. How about when there is a mass present? We can find out by substituting this equation into the equation of motion (above)

$$(0.25) \quad \left(\frac{du}{d\phi} \right)^2 + u^2 = \frac{1}{b^2} + 2GMu^3$$

and saving only terms linear in GM/r_c . We have

$$(0.26) \quad \frac{du}{d\phi} = \frac{1}{r_c} \left(\cos \phi - \frac{GM}{r_c} \sin 2\phi \right)$$

(0.27)

$$\left(\cos \phi - \frac{GM}{r_c} \sin 2\phi \right)^2 + \left(\sin \phi + \frac{3GM}{2r_c} \left(1 + \frac{1}{3} \cos 2\phi \right) \right)^2 = \frac{r_c^2}{b^2} + \frac{2GM}{r_c} \left(\sin \phi + \frac{3GM}{2r_c} \left(1 + \frac{1}{3} \cos 2\phi \right) \right)$$

We can now save only terms up to those involving GM/r_c :

(0.28)

$$\cos^2 \phi - 2 \frac{GM}{r_c} \cos \phi \sin 2\phi + \sin^2 \phi + \frac{3GM}{r_c} \sin \phi \left(1 + \frac{1}{3} \cos 2\phi \right) \approx \frac{r_c^2}{b^2} + \frac{2GM}{r_c} \sin^3 \phi$$

Collecting terms, we get

$$(0.29) \quad \frac{r_c^2}{b^2} \approx 1 + \frac{GM}{r_c} \left[-2 \cos \phi \sin 2\phi + 3 \sin \phi + \sin \phi \cos 2\phi - 2 \sin^3 \phi \right]$$

Using the shorthand notation $s \equiv \sin \phi$ and $c \equiv \cos \phi$, and the identities $\sin 2\phi = 2sc$ and $\cos 2\phi = 1 - 2s^2$, we get

(0.30)

$$-2 \cos \phi \sin 2\phi + 3 \sin \phi + \sin \phi \cos 2\phi - 2 \sin^3 \phi = -4sc^2 + 3s + s - 2s^3 - 2s^3$$

(0.31)

$$= -4s(1 - s^2) + 3s + s - 2s^3 - 2s^3$$

(0.32)

$$= -4s + 4s + 4s^3 - 4s^3$$

(0.33)

$$= 0$$

Thus to this level of approximation, $r_c = b$.

Finally, in our coordinate system the closest approach to the mass occurs at $\phi = \frac{\pi}{2}$ so from 0.24 this distance is

$$(0.34) \quad \frac{1}{r_{min}} = \frac{1}{r_c} \left(1 + \frac{3GM}{2r_c} \left(1 - \frac{1}{3} \right) \right)$$

$$(0.35) \quad = \frac{1}{r_c} \left(1 + \frac{GM}{r_c} \right)$$

Thus

$$(0.36) \quad r_{min} = r_c \left(1 + \frac{GM}{r_c} \right)^{-1}$$

$$(0.37) \quad \approx r_c \left(1 - \frac{GM}{r_c} \right)$$

where the last line uses a Taylor expansion to first order. The closest approach is therefore less than the impact parameter $b = r_c$.

PINGBACKS

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