DEFLECTION OF LIGHT BY A MASS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 13; Problem 13.1.

We've seen how to derive the equations of motion for a photon in a gravitational field, so we can now apply these equations to study the deflection of light as it passes a massive object. The equations of motion are

$$\frac{d\phi}{dt} = \frac{1}{r^2}\left(1-\frac{2GM}{r}\right)b \tag{1}$$

$$\left[\frac{1}{b}\left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{dt}\right]^2 + \frac{1}{r^2}\left(1-\frac{2GM}{r}\right) = \frac{1}{b^2} \tag{2}$$

These equations describe a photon's motion for $r > 2GM$, but when we're discussing the deflection of light as it passes a star such as the sun, typically $r \gg 2GM$. For example, for the Sun, $2GM = 2.954$ km and the radius of the sun is 6.955×10^5 km so a photon grazing the surface of the Sun as it passed has a value of $r/2GM = 4.25\times10^{-6}$. In this limiting case, we can make a few approximations to get an idea of how much light is deflected as it passes a massive object.

Things are a bit easier if we work with the variable $u \equiv 1/r$, so we can use the chain rule to write

$$\begin{aligned}
\frac{dr}{dt} &= \frac{dr}{du}\frac{du}{d\phi}\frac{d\phi}{dt} &(3)\\
&= -\frac{1}{u^2}\frac{du}{d\phi}u^2(1-2GMu)\,b &(4)\\
&= -(1-2GMu)\,b\frac{du}{d\phi} &(5)
\end{aligned}$$

Plugging this into the radial equation of motion above, we get

$$\left(\frac{du}{d\phi}\right)^2+u^2(1-2GMu) = \frac{1}{b^2} \quad (6)$$

$$\left(\frac{du}{d\phi}\right)^2+u^2 = \frac{1}{b^2}+2GMu^3 \quad (7)$$

We can take the derivative of this with respect to ϕ and get

$$2\frac{du}{d\phi}\frac{d^2u}{d\phi^2}+2u\frac{du}{d\phi} = 6GMu^2\frac{du}{d\phi} \quad (8)$$

$$\frac{d^2u}{d\phi^2}+u = 3GMu^2 \quad (9)$$

If $M=0$ (that is, space is flat), this equation reduces to

$$\frac{d^2u}{d\phi^2}+u=0 \quad (10)$$

which has the general solution

$$u=A\sin\phi+B\cos\phi \quad (11)$$

If we orient the coordinate system so that $\phi=\frac{\pi}{2}$ is the angle of closest approach to the origin, then $u=1/r$ is a maximum at that angle so

$$A\cos\frac{\pi}{2}-B\sin\frac{\pi}{2} = 0 \quad (12)$$

$$B = 0 \quad (13)$$

and we can set $A\equiv 1/r_c$ where r_c is the distance of closest approach to the origin:

$$u=\frac{1}{r}=\frac{1}{r_c}\sin\phi \quad (14)$$

This is, in fact, the equation of a straight line in polar coordinates as can be verified by drawing the right-angled triangle with sides r (as the hypotenuse) and $r_c=r\sin\phi$ (the third side is just $r\cos\phi$). From the equation above, with $M=0$, we have

$$\left(\frac{du}{d\phi}\right)^2+u^2=\frac{1}{r_c^2}\left(\cos^2\phi+\sin^2\phi\right)=\frac{1}{r_c^2}=\frac{1}{b^2} \quad (15)$$

Thus $r_c = b$, which is the impact parameter, and gives the distance of closest approach.

Now let's put the mass back in and apply the assumption $r_c \gg 2GM$. In this case, we expect that u will be fairly close to the flat space solution, so we can try a solution of the form

$$u = \frac{1}{r_c}\sin\phi + \frac{w(\phi)}{r_c} \tag{16}$$

where w is some function that should be small for all values of ϕ. Plugging this into the equation of motion above:

$$\frac{d^2u}{d\phi^2} + u = 3GMu^2 \tag{17}$$

$$-\frac{1}{r_c}\sin\phi + \frac{1}{r_c}\frac{d^2w}{d\phi^2} + \frac{1}{r_c}\sin\phi + \frac{w(\phi)}{r_c} = \frac{3GM}{r_c^2}\left(\sin^2\phi + 2w\sin\phi + w^2\right) \tag{18}$$

Because of our assumption $r_c \gg 2GM$, $r_c \gg 3GM$ as well, so if we're saving only up to first-order terms in the 'small' quantities, only the first term on the RHS will contribute. We then get

$$\frac{d^2w}{d\phi^2} + w \approx \frac{3GM}{r_c}\sin^2\phi \tag{19}$$

$$= \frac{3GM}{2r_c}(1 - \cos 2\phi) \tag{20}$$

This differential equation can be solved by assuming a solution of form $w = A + B\cos 2\phi$:

$$-4B\cos 2\phi + A + B\cos 2\phi = \frac{3GM}{2r_c}(1 - \cos 2\phi) \tag{21}$$

from which we get

$$A = \frac{3GM}{2r_c} \tag{22}$$

$$B = \frac{GM}{2r_c} \tag{23}$$

So

$$u = \frac{1}{r_c}\left(\sin\phi + \frac{3GM}{2r_c}\left(1+\frac{1}{3}\cos 2\phi\right)\right) \tag{24}$$

We saw above that in flat space, $r_c = b$, the impact parameter. How about when there is a mass present? We can find out by substituting this equation into the equation of motion (above)

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{1}{b^2} + 2GMu^3 \tag{25}$$

and saving only terms linear in GM/r_c. We have

$$\frac{du}{d\phi} = \frac{1}{r_c}\left(\cos\phi - \frac{GM}{r_c}\sin 2\phi\right) \tag{26}$$

$$\left(\cos\phi - \frac{GM}{r_c}\sin 2\phi\right)^2 + \left(\sin\phi + \frac{3GM}{2r_c}\left(1+\frac{1}{3}\cos 2\phi\right)\right)^2 = \frac{r_c^2}{b^2} + \frac{2GM}{r_c}\left(\sin\phi + \frac{3GM}{2r_c}\left(1+\frac{1}{3}\cos\right.\right. \tag{27}$$

We can now save only terms up to those involving GM/r_c:

$$\cos^2\phi - 2\frac{GM}{r_c}\cos\phi\sin 2\phi + \sin^2\phi + \frac{3GM}{r_c}\sin\phi\left(1+\frac{1}{3}\cos 2\phi\right) \approx \frac{r_c^2}{b^2} + \frac{2GM}{r_c}\sin^3\phi \tag{28}$$

Collecting terms, we get

$$\frac{r_c^2}{b^2} \approx 1 + \frac{GM}{r_c}\left[-2\cos\phi\sin 2\phi + 3\sin\phi + \sin\phi\cos 2\phi - 2\sin^3\phi\right] \tag{29}$$

Using the shorthand notation $s \equiv \sin\phi$ and $c \equiv \cos\phi$, and the identities $\sin 2\phi = 2sc$ and $\cos 2\phi = 1-2s^2$, we get

$$\begin{aligned}
-2\cos\phi\sin 2\phi + 3\sin\phi + \sin\phi\cos 2\phi - 2\sin^3\phi &= -4sc^2 + 3s + s - 2s^3 - 2s^3 && (30)\\
&= -4s\left(1-s^2\right) + 3s + s - 2s^3 - 2s^3 && (31)\\
&= -4s + 4s + 4s^3 - 4s^3 && (32)\\
&= 0 && (33)
\end{aligned}$$

Thus to this level of approximation, $r_c = b$.

Finally, in our coordinate system the closest approach to the mass occurs at $\phi=\frac{\pi}{2}$ so from 24 this distance is

$$\frac{1}{r_{min}} = \frac{1}{r_c}\left(1+\frac{3GM}{2r_c}\left(1-\frac{1}{3}\right)\right) \tag{34}$$

$$= \frac{1}{r_c}\left(1+\frac{GM}{r_c}\right) \tag{35}$$

Thus

$$r_{min} = r_c\left(1+\frac{GM}{r_c}\right)^{-1} \tag{36}$$

$$\approx r_c\left(1-\frac{GM}{r_c}\right) \tag{37}$$

where the last line uses a Taylor expansion to first order. The closest approach is therefore less than the impact parameter $b=r_c$.

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