GRAVITATIONAL LENSING: IMAGE BRIGHTNESS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 13; Problem P13.2.

In gravitational lensing, the brightness of the images can be different from that of the source. To see this, picture the following setup, using variables we introduced in the last post. If you have access to Moore's book, look at Fig. 13.4b.

Put yourself at the observer's location O and look along the line OL towards the lensing mass L. Superimpose a 2-d polar coordinate system with its origin at L and in a plane perpendicular to the observer's line of sight. First, suppose we remove the mass at L so the observer has an unobstructed view of the distant source S. If the source appears larger than a point (that is, it's a galaxy or other extended object) then we can measure its extent in this polar coordinate system by specifying the extent in the radial and circumferential directions. The extent in the radial direction is measured by the range $\Delta\beta$ in the angle β which is the angle between the lines OL and OS (observer to source). The range in the circumferential direction can be expressed as $\Delta\phi$ where ϕ is the angular coordinate in the polar system.

Now put the lensing mass back at position L. Photons passing near L on their way to the observer are bent, but only the radial coordinate is affected, since the bending always occurs towards the mass at L, that is, always in a radial direction. Thus the circumferential range $\Delta\phi$ in each of the two images is the same as in the unobstructed view.

To measure the brightness of an image, we need to calculate the total amount of light that arrives at the observer for that image. This is proportional to the area of each image, which we can calculate by measuring the area in the polar coordinate system occupied by each image.

The actual distance in the radial direction of the unobstructed image is the distance from the observer to the image times the angular extent of that image, that is $D_L\Delta\beta$. The same calculation for the two lensed images gives $D_L\Delta\theta_\pm$. The distance in the circumferential direction is the angle range $\Delta\phi$ times the radius of the circle along which the image lies, which is $D_L\beta$ for the unobstructed image and $D_L\theta_\pm$ for the two lensed images. Therefore, the area of the unobstructed image is $(D_L\Delta\beta)(D_L\beta\Delta\phi)$ and for the lensed images the areas are $(D_L\Delta\theta_\pm)(D_L|\theta_\pm|\Delta\phi)$. (We've add the absolute value since area must be positive and as we've seen, $\theta_{-}<0$.) The ratio of the areas, and hence the ratio of brightnesses, is then

$$\frac{\left(D_{L}\Delta\theta_{\pm}\right)\left(D_{L}\left|\theta_{\pm}\right|\Delta\phi\right)}{\left(D_{L}\Delta\beta\right)\left(D_{L}\beta\Delta\phi\right)}=\frac{\left|\theta_{\pm}\right|\Delta\theta_{\pm}}{\beta\Delta\beta} \tag{1}$$

We can eliminate $\theta_{\pm}$ using the formulas from the last post:

$$\theta_{\pm} = \frac{1}{2}\left(\beta\pm\sqrt{\beta^{2}+4\theta_{E}^{2}}\right) \tag{2}$$

$$\Delta\theta_{\pm} = \frac{1}{2}\left(1\pm\frac{\beta}{\sqrt{\beta^{2}+4\theta_{E}^{2}}}\right)\Delta\beta \tag{3}$$

Plugging these into the ratio we get

$$\frac{\left|\theta_{\pm}\right|\Delta\theta_{\pm}}{\beta\Delta\beta}=\frac{1}{4}\left|\left(1\pm\frac{\sqrt{\beta^{2}+4\theta_{E}^{2}}}{\beta}\right)\left(1\pm\frac{\beta}{\sqrt{\beta^{2}+4\theta_{E}^{2}}}\right)\right| \tag{4}$$

$$=\frac{1}{4}\left|\left(2\pm\left[\frac{\sqrt{\beta^{2}+4\theta_{E}^{2}}}{\beta}+\frac{\beta}{\sqrt{\beta^{2}+4\theta_{E}^{2}}}\right]\right)\right| \tag{5}$$

For $x>0$ the function $f(x)=x+1/x$ has a minimum at $f'(x)=1-1/x^{2}=0$ or $x=1$. The minimum value is $f(1)=2$, so the expression in square brackets above is always ≥ 2. Therefore, we can eliminate the absolute value signs by writing

$$\frac{\left|\theta_{\pm}\right|\Delta\theta_{\pm}}{\beta\Delta\beta}=\frac{1}{4}\left(\frac{\sqrt{\beta^{2}+4\theta_{E}^{2}}}{\beta}+\frac{\beta}{\sqrt{\beta^{2}+4\theta_{E}^{2}}}\pm 2\right) \tag{6}$$

As an example, suppose we have a galaxy positioned between a distant quasar and the Earth such that $\beta=\theta_{E}/2$. In that case, the two images are at angles

$$\theta_{\pm}=\frac{\theta_{E}}{4}\left(1\pm\sqrt{17}\right)=1.28\theta_{E},\ -0.78\theta_{E} \tag{7}$$

The relative brightness of each image is

$$\frac{|\theta_{\pm}|\Delta\theta_{\pm}}{\beta\Delta\beta}=\frac{1}{4}\left(\sqrt{17}+\frac{1}{\sqrt{17}}\pm 2\right)=1.59,\ 0.59 \tag{8}$$

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