GRAVITATIONAL LENSING: LARGE ANGLES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 13; Problem P13.4.

The formula for the image angles in gravitational lensing is

$$\theta_{\pm}=\frac{1}{2}\left(\beta\pm\sqrt{\beta^{2}+4\theta_{E}^{2}}\right) \tag{1}$$

where θ_E is the angle subtended by the radius of the Einstein ring when the source object is directly behind the lensing object (that is, when $\beta=0$):

$$\theta_{E}\equiv\sqrt{D_{LS}\frac{4GM}{D_{L}D_{S}}} \tag{2}$$

Notice that θ_E depends on the distances from the observer to the source and lens, and on the mass of the lens, but not on their relative orientation (although it does assume that the angles involved are small). It is therefore possible to investigate what happens when β becomes large relative to θ_E, while still keeping both angles quite small. In this case, we get

$$\begin{aligned}
\theta_{\pm} &= \frac{\beta}{2}\left(1\pm\sqrt{1+4\frac{\theta_{E}^{2}}{\beta^{2}}}\right) && (3)\\
&\approx \frac{\beta}{2}\left(1\pm\left(1+2\frac{\theta_{E}^{2}}{\beta^{2}}\right)\right) && (4)\\
\theta_{+} &\approx \beta+\frac{\theta_{E}^{2}}{\beta} && (5)\\
\theta_{-} &\approx -\frac{\theta_{E}^{2}}{\beta} && (6)
\end{aligned}$$

The outer angle θ_+ tends toward β, the angle to the source in the absence of the lens, while θ_- tends to zero. Not surprisingly, the further the source strays from being directly behind the lens, the smaller the effect of the lens.

The relative brightness of the images is given by

$$\frac{I_{\pm}}{I_s} = \frac{1}{4}\left(\frac{\sqrt{\beta^2+4\theta_E^2}}{\beta} + \frac{\beta}{\sqrt{\beta^2+4\theta_E^2}} \pm 2\right) \tag{7}$$

where $I_{\pm}$ are the brightnesses of the lensed images and I_s is the brightness of the unlensed source. We can write this as

$$\frac{I_{\pm}}{I_s} = \frac{1}{4}\left(\left(1+4\frac{\theta_E^2}{\beta^2}\right)^{1/2} + \left(1+4\frac{\theta_E^2}{\beta^2}\right)^{-1/2} \pm 2\right) \tag{8}$$

Doing a Taylor expansion and keeping only the first non-zero term we get

$$\frac{I_+}{I_s} = 1+\frac{\theta_E^4}{\beta^4} \tag{9}$$

$$\frac{I_-}{I_s} = \frac{\theta_E^4}{\beta^4} \tag{10}$$

Thus the brightness of the outer image tends towards that of the unlensed source, while the inner image fades to zero.

We can work out these values for starlight deflected by the sun. Since any star is much further from the Earth than the Sun, we can take $D_{LS} = D_S$ so the Einstein ring angle becomes

$$\theta_E = \sqrt{\frac{4GM}{D_L}} \tag{11}$$

For the sun, $GM = 1.477$ km and $D_L = 1.496 \times 10^8$ km so $\theta_E = 1.987 \times 10^{-4}$ radians. The angle β to the unlensed star, assuming its light just grazes the sun's surface, is then around

$$\beta = \frac{R_{sun}}{D_L} = \frac{6.955 \times 10^5}{1.496 \times 10^8} = 4.649 \times 10^{-3} \text{ radians} \tag{12}$$

Thus in the solar case, $\theta_E/\beta = 0.0427$ which isn't *very* small, but probably small enough to get an estimate of the angles and brightnesses involved.

The angles of the two images are

$$\theta_+ \approx \beta + \frac{\theta_E^2}{\beta} = \frac{R_{sun}}{D_L} + \frac{4GM}{R_{sun}} \tag{13}$$

$$\theta_- \approx -\frac{\theta_E^2}{\beta} = -\frac{4GM}{R_{sun}} \tag{14}$$

The deflection of the outer image is therefore

$$\frac{4GM}{R_{sun}} = 8.495 \times 10^{-6} \text{ radians} = 1.752'' \tag{15}$$

This agrees with the earlier result.

The inner image occurs at

$$|\theta_-| = \frac{4GM}{R_{sun}} = 1.752'' \tag{16}$$

Since the sun's radius subtends an angle of around 16 minutes of arc (that is, $\beta = 15.982$ min) this inner image is behind the sun's disk and is therefore invisible.