DELAY OF LIGHT PASSING A MASS: SHAPIRO DELAY

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 13; Problem P13.7.

In addition to being deflected when it passes a mass, photons also experience a delay (relative to the travel time in the absence of the deflecting mass) known as the Shapiro delay, after Irwin Shapiro who first observed the effect in the 1960s. Crudely speaking, the delay is due to the curvature of space-time resulting in the photons having to travel a longer path, although of course the actual reason is a bit more subtle, being related to the revised notions of space and time in the Schwarzschild metric.

Our starting point for calculating the delay in the travel time is the photon radial equation of motion:

$$\left[\frac{1}{b}\left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{dt}\right]^2+\frac{1}{r^2}\left(1-\frac{2GM}{r}\right)=\frac{1}{b^2} \tag{1}$$

The closest approach of the photon to the mass occurs when r is a minimum at $r=r_0$, and thus when $dr/dt=0$. Thus

$$\frac{1}{r_0^2}\left(1-\frac{2GM}{r_0}\right) = \frac{1}{b^2} \tag{2}$$

$$b = r_0\left(1-\frac{2GM}{r_0}\right)^{-1/2} \tag{3}$$

We can express the original differential equation in terms of r_0 by eliminating b.

$$\frac{1}{r_0^2}\left(1-\frac{2GM}{r_0}\right)\left[\left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{dt}\right]^2+\frac{1}{r^2}\left(1-\frac{2GM}{r}\right)=\frac{1}{r_0^2}\left(1-\frac{2GM}{r_0}\right) \tag{4}$$

$$\left[\left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{dt}\right]^2=1-\frac{r_0^2}{r^2}\frac{1-2GM/r}{1-2GM/r_0} \tag{5}$$

$$\frac{dr}{dt}=\left(1-\frac{2GM}{r}\right)\frac{1}{r}\sqrt{r^2-r_0^2\frac{1-2GM/r}{1-2GM/r_0}} \tag{6}$$

Trying to integrate this is impossible as it looks (except numerically), so we need to simplify it by making a few approximations. As usual, we can assume that $2GM \ll r_0 \le r$ (that is, the photon never gets close to the Schwarzschild radius $2GM$). As a first attempt, we might just ignore these terms (that is, take $2GM/r_0 \approx 0$). If we do that, we get

$$\begin{aligned}
\frac{dr}{dt} &= \frac{\sqrt{r^2-r_0^2}}{r} && (7)\\
t-t_0 &= \int_{r_0}^{r}\frac{r'dr'}{\sqrt{(r')^2-r_0^2}} && (8)\\
t-t_0 &= \sqrt{r^2-r_0^2} && (9)
\end{aligned}$$

where t_0 is the time at which the photon is at the closest approach r_0. Thus $t-t_0$ is the time to travel to or from the point of closest approach to some point r in flat space (since we've taken $M=0$), so this obviously can't include the delay factor. We need a better approximation, and one that is hopefully still integrable.

We'll introduce the variable $u \equiv r/r_0 \ge 1$ and rewrite the equation above:

$$\frac{dr}{dt} = \left(1-\frac{2GM}{r_0u}\right)\frac{1}{r_0u}\sqrt{r_0^2u^2 - r_0^2\frac{1-2GM/r_0u}{1-2GM/r_0}} \tag{10}$$

$$= \left(1-\frac{2GM}{r_0u}\right)\frac{1}{u}\sqrt{u^2 - \frac{1-2GM/r_0u}{1-2GM/r_0}} \tag{11}$$

$$= \left(1-\frac{2GM}{r_0u}\right)\sqrt{1 - \frac{1}{u^2}\frac{1-2GM/r_0u}{1-2GM/r_0}} \tag{12}$$

Since we're assuming $2GM/r_0 \ll 1$ and $u \geq 1$, $\frac{2GM}{r_0u} \ll 1$ as well, so we can expand the operand of the square root in a Taylor series and keep only the first order term. We define $\gamma \equiv 2GM/r_0$ to simplify the notation.

$$1-\frac{1}{u^2}\frac{1-\gamma/u}{1-\gamma} \approx 1-\frac{1}{u^2}\left(1-\frac{\gamma}{u}\right)(1+\gamma) \tag{13}$$

$$\approx 1-\frac{1}{u^2}\left(1+\gamma-\frac{\gamma}{u}\right) \tag{14}$$

$$= \frac{u^2-1-\gamma+\gamma/u}{u^2} \tag{15}$$

$$= \frac{u^2-1+(1-u)\gamma/u}{u^2} \tag{16}$$

$$= \frac{u^2-1-(u-1)(u+1)\gamma/u(u+1)}{u^2} \tag{17}$$

$$= \frac{\left(u^2-1\right)}{u^2}\left(1-\frac{\gamma}{u(u+1)}\right) \tag{18}$$

Therefore

$$\frac{dr}{dt} = r_0\frac{du}{dt} \tag{19}$$

$$= \frac{\sqrt{u^2-1}}{u}\left(1-\frac{2GM}{r_0u}\right)\sqrt{1-\frac{2GM}{r_0u(u+1)}} \tag{20}$$

Thus we're now faced with

$$t-t_0 = \int_1^u \frac{r_0u\,du}{\sqrt{u^2-1}\left(1-\frac{2GM}{r_0u}\right)\sqrt{1-\frac{2GM}{r_0u(u+1)}}} \tag{21}$$

(Note: the minus sign in the last square root is correct; the plus sign given by Moore in equation 13.27 is wrong. See the list of errata in the book.)

Needless to say, this integral still can't be done, but we can approximate yet again by noting that

$$\left(1-\frac{2GM}{r_0 u}\right)^{-1} \approx 1+\frac{2GM}{r_0 u} \tag{22}$$

$$\left(1-\frac{2GM}{r_0 u(u+1)}\right)^{-1/2} \approx 1+\frac{GM}{r_0 u(u+1)} \tag{23}$$

Thus to first order, we now have

$$\frac{t-t_0}{r_0}=\int_1^u \frac{udu}{\sqrt{u^2-1}}+\frac{2GM}{r_0}\int_1^u \frac{du}{\sqrt{u^2-1}}+\frac{GM}{r_0}\int_1^u \frac{du}{(u+1)\sqrt{u^2-1}} \tag{24}$$

These integrals can be done (using software or tables), and we get (substituting back for $u=r/r_0$):

$$t-t_0=\sqrt{r^2-r_0^2}+2GM\ln\frac{r+\sqrt{r^2-r_0^2}}{r_0}+GM\sqrt{\frac{r-r_0}{r+r_0}} \tag{25}$$

The first term in this answer is the flat space term, so the latter two terms represent the delay. Both these terms are non-negative (since $r\ge r_0$ the argument of the logarithm is ≥ 1 so it is non-negative).

As an example, we can work out the delay for a photon travelling from Venus to Earth when Venus is on the opposite side of the sun from Earth. The radius of Venus's orbit is 0.723 AU (astronomical units) and the radius of the Earth's orbit is 1 AU. If the photon just grazes the surface of the sun, r_0 is the sun's radius which is 6.955×10^5 km$/1.496\times 10^8$ km $=4.652\times 10^{-3}$ AU. GM for the sun is 1.477 km.

The delays in travelling from Venus to the Earth are then (divide by the speed of light to convert from km to sec):

$$t_0-t_{Venus} = 18.42\text{ km}=61.4\times 10^{-6}\text{ sec} \tag{26}$$

$$t_{Earth}-t_0 = 19.382\text{ km}=64.6\times 10^{-6}\text{ sec} \tag{27}$$

$$t_{Earth}-t_{Venus} = 126\times 10^{-6}\text{ sec} \tag{28}$$

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