SHAPIRO DELAY: THE TWIN QUASAR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 13; Problem P13.8.

Since the light producing the two images of a distant source when lensed by a closer mass travel different distances (unless the source is directly behind the lens), the Shapiro delay in the travel time can be different for the two images. We can get an idea of the difference in the delays by the following argument.

To set up the problem, we need to define a few parameters. Reference to Moore's Fig. 13.9 will be helpful here (it's a bit too complex for me to reproduce), but I'll try to describe the parameters so you can draw a diagram if you follow along.

We define three points: O for the observer on Earth, L for the lens and S for the distant source. The distance from O to L is D_L and the distance from L to S is D_{LS} as before. We assume that O, L and S are almost, but not quite, in a straight line. The angle between OL and OS is β, θ_+ is the larger of the two angles to an image (the brighter one) of S as seen by O, and $|\theta_-|$ is the angle to the smaller and dimmer image (absolute values are needed as $\theta_- < 0$). We can regard $\theta_\pm$ as the angles to the points of closest approach of photons to L as they pass on either side of L.

Now look at the setup from the point of view of an observer on S. We can define the angles $\alpha_\pm$ as the angles between the line SO and the points of closest approach of photons leaving S as they pass L. Notice that $\alpha_\pm$ are qualitatively different from $\theta_\pm$, since the $\alpha_\pm$ angles are measured from the line connecting O and S, while the $\theta_\pm$ angles are measured from the line connecting O and L.

Returning our focus to the observer O on Earth, the angles between the line OS and the points of closest approach are $\theta_+ - \beta$ to the image further from L and $|\theta_-| + \beta$ to the image nearer to L.

If you've managed to construct the diagram properly (or just looked at Moore's Fig. 13.9) then, since the angles are all very small, we have

$$D_{LS}\alpha_+ \approx D_L(\theta_+ - \beta) \tag{1}$$

$$D_{LS}\alpha_- \approx D_L(|\theta_-| + \beta) \tag{2}$$

That is, the distance swept out by the line D_{LS} through an angle α_+ is the same as that swept out by D_L through an angle $\theta_+ - \beta$, and similarly for the angles on the other side of L.

The angles $\theta_\pm$ are from the formula

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right) \tag{3}$$

so we have

$$\theta_+ - \beta = \sqrt{\beta^2 + 4\theta_E^2} - \frac{\beta}{2} = \theta_0 - \frac{\beta}{2} \tag{4}$$

$$|\theta_-| + \beta = \sqrt{\beta^2 + 4\theta_E^2} + \frac{\beta}{2} = \theta_0 + \frac{\beta}{2} \tag{5}$$

where $\theta_0 = \sqrt{\beta^2 + 4\theta_E^2}$.

We can now write $\alpha_\pm$ as

$$\alpha_+ = \frac{D_L}{D_{LS}}(\theta_+ - \beta) = \frac{D_L}{D_{LS}}\left(\theta_0 - \frac{\beta}{2}\right) \tag{6}$$

$$\alpha_- = \frac{D_L}{D_{LS}}(|\theta_-| + \beta) = \frac{D_L}{D_{LS}}\left(\theta_0 + \frac{\beta}{2}\right) \tag{7}$$

The path lengths of photons contributing to the two images are then:

$$\ell_+ = \frac{D_{LS}}{\cos\alpha_+} + \frac{D_L}{\cos(\theta_+ - \beta)} \tag{8}$$

$$\ell_- = \frac{D_{LS}}{\cos\alpha_-} + \frac{D_L}{\cos(|\theta_-| + \beta)} \tag{9}$$

We can't just approximate all the cosines by 1, since that would give $\ell_+ = \ell_-$ which doesn't do us much good. Therefore, we need to keep the square term and approximate $\cos x \approx 1 - \frac{1}{2}x^2$. We then get, using $1/\left(1 - x^2/2\right) \approx 1 + \frac{x^2}{2}$:

$$\ell_+ \approx D_{LS}\left(1 + \frac{D_L^2}{2D_{LS}^2}\left(\theta_0 - \frac{\beta}{2}\right)^2\right) + D_L\left(1 + \frac{1}{2}\left(\theta_0 - \frac{\beta}{2}\right)^2\right) \tag{10}$$

$$\ell_- \approx D_{LS}\left(1 + \frac{D_L^2}{2D_{LS}^2}\left(\theta_0 + \frac{\beta}{2}\right)^2\right) + D_L\left(1 + \frac{1}{2}\left(\theta_0 + \frac{\beta}{2}\right)^2\right) \tag{11}$$

Taking the difference, all terms except those in $\theta_0\beta$ will cancel, and we have

$$\ell_- - \ell_+ = \frac{D_L^2}{D_{LS}}\theta_0\beta + D_L\theta_0\beta = D_L\left(1+\frac{D_L}{D_{LS}}\right)\theta_0\beta \tag{12}$$

We can apply this to the twin quasar we looked at earlier as an example of the double image. In this case, the numbers are $D_L = 3.7\times 10^9$ ly and $D_S = 8.7\times 10^9$ ly so $D_{LS} = D_S - D_L = 5.0\times 10^9$ ly. We need θ_E in radians, so we have

$$\theta_E = \sqrt{5}\text{''} = \frac{\sqrt{5}}{3600}\frac{\pi}{180} = 1.084\times 10^{-5}\text{ radians} \tag{13}$$

We are also given $\beta = 1.8\theta_E = 1.95\times 10^{-5}$ radians, so

$$\theta_0 = \sqrt{\beta^2 + 4\theta_E^2} = 2.917\times 10^{-5} \tag{14}$$

So

$$\ell_- - \ell_+ = D_L\left(1+\frac{D_L}{D_{LS}}\right)\theta_0\beta = 3.66\text{ light years} \tag{15}$$

Thus the actual time delay between the two branches is 3.66 years or 1337 days. This can be checked by observation, since the quasar varies in brightness so we can measure the time between the changes in brightness of the two images. The actual measured time is 417 ± 3 days so the agreement isn't great, but we did make a lot of approximations along the way so this isn't too surprising.