EVENT HORIZON: DISTANCE FROM EXTERNAL RADIUS UP TO R = 2GM

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 14; Box 14.1.

It's fairly obvious from looking at the Schwarzschild metric that something odd happens at $r = 2GM$ (known as the event horizon):

(1)
$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2$$

Both the time and radial components change sign, with the time component going through zero and the radial component through infinity. There are two possible reasons why something seems to be happening here. The first possibility is that there is something genuinely odd about the geometry at that radius, and no matter what coordinate system we use to describe it, this pathological behaviour will not go away. One of the assumptions that went into relativity is that if we examine any part of space-time on a small enough scale, it can be approximated by flat space (this is the idea behind using manifolds to describe space-time). If this isn't true, then obviously the physics we derive from that assumption cannot be valid either. A good analogy in two dimensions that of calculating the derivative of a function. The derivative is based on the idea that if we look at a curve closely enough, it can be approximated by a straight line so that the derivative can be defined as the slope of that line. This assumption is false for any function which makes a sharp bend (as happens at the vertex of a triangle) and at such points the derivative cannot be defined uniquely. If something like this is happening in four dimensional space-time, then the Schwarzschild metric cannot describe it properly.

The other possibility is that the coordinate system we are using breaks down at certain points. In this case, choosing a different coordinate system can resolve the problem. Again in two dimensions, if we use polar coordinates (r,θ) to describe the plane, the origin is defined by $r = 0$ but at this point we cannot assign a unique value to θ. There is nothing special about the origin compared to any other point on the plane, and if we choose a different coordinate system (such as rectangular), the problem goes away.

One indication that the problem might be related to the coordinate system is that the space-time interval between two events that differ only in their r coordinates is finite, even though the g_{rr} component of the metric goes to infinity at $r = 2GM$. From the metric above, we get

$$\Delta s = \int_{2GM}^{R} \frac{dr}{\sqrt{1-\frac{2GM}{r}}} \tag{2}$$

Plugging the indefinite integral into Maple we get

$$\int \frac{dr}{\sqrt{1-\frac{2GM}{r}}} = \sqrt{r(r-2GM)} + GM\ln\left(r - GM + \sqrt{r(r-2GM)}\right) \tag{3}$$

so between the limits given, we get

$$\Delta s = R\sqrt{1-\frac{2GM}{R}} + GM\ln\left(R - GM + R\sqrt{1-\frac{2GM}{R}}\right) - GM\ln(GM) \tag{4}$$

$$= R\sqrt{1-\frac{2GM}{R}} + GM\ln\left[\frac{R - GM + R\sqrt{1-\frac{2GM}{R}}}{GM}\right] \tag{5}$$

Although this form seems suitable for calculation, we can convert it into the form in Moore's book by noting that

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \tag{6}$$

With $x = \sqrt{1-\frac{2GM}{R}}$ we get

$$\frac{1+\sqrt{1-\frac{2GM}{R}}}{1-\sqrt{1-\frac{2GM}{R}}} = \frac{1+\sqrt{1-\frac{2GM}{R}}}{1-\sqrt{1-\frac{2GM}{R}}}\frac{1+\sqrt{1-\frac{2GM}{R}}}{1+\sqrt{1-\frac{2GM}{R}}} \tag{7}$$

$$= \frac{1+2\sqrt{1-\frac{2GM}{R}}+1-\frac{2GM}{R}}{2GM/R} \tag{8}$$

$$= \frac{1-\frac{GM}{R}+\sqrt{1-\frac{2GM}{R}}}{GM/R} \tag{9}$$

$$= \frac{R-GM+R\sqrt{1-\frac{2GM}{R}}}{GM} \tag{10}$$

Thus we get the final form:

$$\Delta s = R\sqrt{1-\frac{2GM}{R}}+2GM\tanh^{-1}\sqrt{1-\frac{2GM}{R}} \tag{11}$$

Evaluating the logarithm formula for $R=3GM$ we get

$$\Delta s = GM\left(\sqrt{3}+\ln\left(2+\sqrt{3}\right)\right) = 3.049GM \tag{12}$$

Thus the distance from an exterior radial coordinate up to $r=2GM$ is finite even though $g_{rr}\to\infty$ there.

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