

## EVENT HORIZON: PROPER TIME TO FALL INTO $R = 0$

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 14; Box 14.2.

Continuing our exploration of the Schwarzschild metric and its behaviour at  $r = 2GM$  (known as the event horizon), we'll look here at the proper time an object measures as it crosses the event horizon. The metric is:

$$(0.1) \quad ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

Consider an object starting at rest at a radial coordinate  $r = R$  and falling radially in towards the event horizon. To work out its proper time, we start with the equation of motion for  $r$ :

$$(0.2) \quad \frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2} \frac{\ell^2}{r^2} - GM \left(\frac{1}{r} + \frac{\ell^2}{r^3}\right) = \frac{1}{2} (e^2 - 1)$$

Since the object is at rest initially,  $dr/d\tau = 0$  and its angular momentum is  $\ell = 0$  so we get for its energy

$$(0.3) \quad e = \sqrt{1 - \frac{2GM}{R}}$$

The energy  $e$  is a conserved quantity, so the equation of motion becomes

$$(0.4) \quad \frac{dr}{d\tau} = \sqrt{2GM} \sqrt{\frac{1}{r} - \frac{1}{R}}$$

We can integrate this to find the proper time that elapses as the object falls from  $r = R$  to  $r = 0$ :

$$(0.5) \quad \int_R^0 \frac{-dr}{\sqrt{\frac{1}{r} - \frac{1}{R}}} = \sqrt{2GM} \Delta\tau$$

(the minus sign in the integrand accounts for the fact that  $r$  is decreasing).

Using Maple, the integral is

$$(0.6) \quad \int \frac{-dr}{\sqrt{\frac{1}{r} - \frac{1}{R}}} = \frac{\sqrt{R}}{2} \left[ 2\sqrt{r(R-r)} + R \arctan \left( \frac{R-2r}{2\sqrt{r(R-r)}} \right) \right]$$

Plugging in the limits, we get

$$(0.7) \quad \sqrt{2GM}\Delta\tau = \frac{\pi R^{3/2}}{2}$$

$$(0.8) \quad \Delta\tau = \frac{\pi R^{3/2}}{\sqrt{8GM}}$$

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