EVENT HORIZON: PROPER TIME TO FALL INTO R = 0

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 14; Box 14.2.

Continuing our exploration of the Schwarzschild metric and its behaviour at $r=2GM$ (known as the event horizon), we'll look here at the proper time an object measures as it crosses the event horizon. The metric is:

(1)
$$ds^2=-\left(1-\frac{2GM}{r}\right)dt^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2$$

Consider an object starting at rest at a radial coordinate $r=R$ and falling radially in towards the event horizon. To work out its proper time, we start with the equation of motion for r:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+\frac{1}{2}\frac{\ell^2}{r^2}-GM\left(\frac{1}{r}+\frac{\ell^2}{r^3}\right)=\frac{1}{2}\left(e^2-1\right) \tag{2}$$

Since the object is at rest initially, $dr/d\tau=0$ and its angular momentum is $\ell=0$ so we get for its energy

$$e=\sqrt{1-\frac{2GM}{R}} \tag{3}$$

The energy e is a conserved quantity, so the equation of motion becomes

$$\frac{dr}{d\tau}=\sqrt{2GM}\sqrt{\frac{1}{r}-\frac{1}{R}} \tag{4}$$

We can integrate this to find the proper time that elapses as the object falls from $r=R$ to $r=0$:

$$\int_R^0\frac{-dr}{\sqrt{\frac{1}{r}-\frac{1}{R}}}=\sqrt{2GM}\Delta\tau \tag{5}$$

(the minus sign in the integrand accounts for the fact that r is decreasing).

Using Maple, the integral is

$$\int\frac{-dr}{\sqrt{\frac{1}{r}-\frac{1}{R}}}=\frac{\sqrt{R}}{2}\left[2\sqrt{r\left(R-r\right)}+R\arctan\left(\frac{R-2r}{2\sqrt{r\left(R-r\right)}}\right)\right] \tag{6}$$

Plugging in the limits, we get

$$\sqrt{2GM}\Delta\tau = \frac{\pi R^{3/2}}{2} \tag{7}$$

$$\Delta\tau = \frac{\pi R^{3/2}}{\sqrt{8GM}} \tag{8}$$

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