FALLING OBJECT OBSERVED NEAR THE EVENT HORIZON

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 14; Problem 14.3.

We can use the techniques we derived earlier to find out what a stationary observer sees when watching an object falling towards the event horizon. Recall that we started with an orthonormal coordinate basis for the observer in locally flat space and then transformed the basis vectors into the global Schwarzschild metric. We got the result:

$$\mathbf{o}_t' = \left[\left(1-\frac{2GM}{R}\right)^{-1/2},0,0,0\right] \tag{0.1}$$

$$\mathbf{o}_x' = \left[0,0,0,\frac{1}{R\sin\theta}\right] \tag{0.2}$$

$$\mathbf{o}_y' = \left[0,0,-\frac{1}{R},0\right] \tag{0.3}$$

$$\mathbf{o}_z' = \left[0,\sqrt{1-\frac{2GM}{R}},0,0\right] \tag{0.4}$$

Here R is the position of the stationary observer, the components are listed in the order $[t,r,\theta,\phi]$ for each vector, and coordinate axes are assumed to be aligned so that x, y and z point along the local ϕ, $-\theta$ and r directions respectively.

To use this to find the velocity of the falling object as viewed by the observer, we need the object's four-velocity $\mathbf{u}$. In the flat frame, the components of $\mathbf{u}$ are given by $u^i=\mathbf{o}_i\cdot\mathbf{u}$ for $i=x$, y or z and $u^t=-\mathbf{o}_t\cdot\mathbf{u}$. The scalar product $\mathbf{o}_i\cdot\mathbf{u}$ is invariant under a change of coordinates, so it must have the same value in the Schwarzschild system, so the flat space coordinates can be calculated in the Schwarzschild system as $u^i=\mathbf{o}_i'\cdot\mathbf{u}'$ for $i=x$, y or z and $u^t=-\mathbf{o}_t'\cdot\mathbf{u}'$. However, we know the components of $\mathbf{u}'$ since they are given by the equations of motion. In general, we have, for motion in the equatorial plane where $\theta=\pi/2$:

$$u'_t = \frac{dt}{d\tau} = e\left(1 - \frac{2GM}{R}\right)^{-1} \tag{0.5}$$

$$u'_z = \frac{dr}{d\tau} = \pm\sqrt{e^2 - \left(1 - \frac{2GM}{R}\right)\left(1 + \frac{\ell^2}{R^2}\right)} \tag{0.6}$$

$$u'_x = \frac{d\phi}{d\tau} = \frac{\ell}{R^2} \tag{0.7}$$

$$u'_y = \frac{d\theta}{d\tau} = 0 \tag{0.8}$$

For an object falling radially inwards, $dr/d\tau < 0$ and $\ell = 0$ and these equations reduce to

$$u'_t = \frac{dt}{d\tau} = e\left(1 - \frac{2GM}{R}\right)^{-1} \tag{0.9}$$

$$u'_z = \frac{dr}{d\tau} = -\sqrt{e^2 - \left(1 - \frac{2GM}{R}\right)} \tag{0.10}$$

$$u'_x = \frac{d\phi}{d\tau} = 0 \tag{0.11}$$

$$u'_y = \frac{d\theta}{d\tau} = 0 \tag{0.12}$$

The velocity components as measured by our stationary observer are

$$v_i = \frac{u^i}{u^t} = \frac{\mathbf{o}'_i \cdot \mathbf{u}'}{-\mathbf{o}'_t \cdot \mathbf{u}'} \tag{0.13}$$

and we now have all the machinery in place to calculate these components. Using the Schwarzschild metric, we get

$$u^t = -\mathbf{o}'_t \cdot \mathbf{u}' \tag{0.14}$$

$$= -g_{tt}\left(1-\frac{2GM}{R}\right)^{-1/2} e\left(1-\frac{2GM}{R}\right)^{-1} \tag{0.15}$$

$$= \left(1-\frac{2GM}{R}\right)\left(1-\frac{2GM}{R}\right)^{-1/2} e\left(1-\frac{2GM}{R}\right)^{-1} \tag{0.16}$$

$$= e\left(1-\frac{2GM}{R}\right)^{-1/2} \tag{0.17}$$

$$u^z = \mathbf{o}'_z \cdot \mathbf{u}' \tag{0.18}$$

$$= g_{rr}\sqrt{1-\frac{2GM}{R}}\left(-\sqrt{e^2-\left(1-\frac{2GM}{R}\right)}\right) \tag{0.19}$$

$$= \left(1-\frac{2GM}{R}\right)^{-1}\sqrt{1-\frac{2GM}{R}}\left(-\sqrt{e^2-\left(1-\frac{2GM}{R}\right)}\right) \tag{0.20}$$

$$= -\left(1-\frac{2GM}{R}\right)^{-1/2}\sqrt{e^2-\left(1-\frac{2GM}{R}\right)} \tag{0.21}$$

$$u^x = u^y = 0 \tag{0.22}$$

Thus the only non-zero velocity component is

$$v_z = -\frac{1}{e}\sqrt{e^2-\left(1-\frac{2GM}{R}\right)} \tag{0.23}$$

and the square of the velocity is

$$v_{obs}^2 = v_z^2 = 1-\frac{1}{e^2}\left(1-\frac{2GM}{R}\right) \tag{0.24}$$

This tends to 1 (the speed of light) as $R \to 2GM$, that is, as the observer gets closer to the event horizon. The actual observed velocity depends on the falling object's energy e (this is the energy at infinity). Note also that this formula breaks down for $R < 2GM$ since then $v_{obs}^2 > 1$ which isn't allowed.

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