LIGHT CONES NEAR THE EVENT HORIZON

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 14; Problem 14.4.

For an object moving in the Schwarzschild metric, its proper time interval is given by

$$d\tau^2 = -ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{1}$$

If the object is moving radially, then $d\theta = d\phi = 0$ and since $d\tau$ must be a real number, we must have $d\tau^2 \geq 0$ so

$$\left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 \geq 0 \tag{2}$$

By taking the equality in this formula, we can find the equation of the light cone for various values of r. In general, we have

$$\left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2GM}{r}\right)^{-2} \tag{3}$$

For various values of r, we get

r	$\left(\frac{dt}{dr}\right)^2$	$\frac{dt}{dr}$
$4GM$	4	± 2
$3GM$	9	± 3
$\frac{5}{2}GM$	25	± 5
$\frac{3}{2}GM$	9	± 3
GM	1	± 1
$\frac{1}{2}GM$	$\frac{1}{9}$	$\pm\frac{1}{3}$

To convert these results into plots of the light cone, we must remember that if $r > 2GM$, increasing proper time corresponds to increasing t, while for $r < 2GM$, space and time swap round, so increasing proper time corresponds to *decreasing* r. If we plot the light cone on a 2-d graph of t versus r, then for $r > 2GM$, the light cone opens upwards and gets progressively narrower as r approaches $2GM$ from above.

For $r < 2GM$, the light cone opens to the left, starting with a very wide (effectively 180°) angle between the two sides, and then getting narrower as $r \to 0$, with the cone becoming a spike that points to the left along the r axis.