BLACK HOLES: ARE THEY REALLY BLACK?

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 14; Problem 14.8.

To see if a black hole really does appear black, let's look at how a photon emitted near the event horizon appears to an observer far from the black hole. This is essentially the inverse of the problem we considered earlier, in which we looked at the Doppler shift of a photon heading towards the black hole from infinity (see end of this post). Using the same derivation as in that post, we start with the equations for the photon's momentum:

$$p^t = E_\infty\left(1-\frac{2GM}{r}\right)^{-1}\frac{dt}{dt} = E_\infty\left(1-\frac{2GM}{r}\right)^{-1} \tag{0.1}$$

$$p^r = E_\infty\left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{dt} \tag{0.2}$$

$$= \pm E_\infty\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}} \tag{0.3}$$

$$p^\theta = 0 \tag{0.4}$$

$$p^\phi = E_\infty\left(1-\frac{2GM}{r}\right)^{-1}\frac{d\phi}{dt} \tag{0.5}$$

$$= E_\infty\frac{b}{r^2} \tag{0.6}$$

where E_∞ is the photon's energy at infinity.

If the photon is moving radially outwards from the black hole, then the impact parameter is $b=0$, and the photon's radial momentum component to be $p_r=+E_\infty$ (positive, since the photon is heading away from the black hole in the direction of increasing r). The energy E_e of the photon in the frame of the emitter (a laser, say), is $E_e=-\mathbf{o}_t\cdot\mathbf{p}$ where $\mathbf{o}_t$ is the t basis vector of the locally flat frame. In the Schwarzschild frame, this has components

$$\mathbf{o}_t = \left[\left(1-\frac{2GM}{r}\right)^{-1}, -\sqrt{\frac{2GM}{r}}, 0, 0\right] \tag{0.7}$$

Therefore

(0.8)

$$E_e = -\mathbf{o}_t \cdot \mathbf{p}$$

(0.9)

$$= E_\infty \left[\left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-1}\left(1-\frac{2GM}{r}\right)^{-1} - \left(1-\frac{2GM}{r}\right)^{-1}\left(-\sqrt{\frac{2GM}{r}}\right)\right]$$

(0.10)

$$= E_\infty \left(1-\frac{2GM}{r}\right)^{-1}\left(1+\sqrt{\frac{2GM}{r}}\right)$$

(0.11)

$$= \frac{E_\infty}{1-\sqrt{\frac{2GM}{r}}}$$

or, since in this case it is E_e that is known:

$$E_\infty = \left(1-\sqrt{\frac{2GM}{r}}\right)E_e \tag{0.12}$$

Since $E_\infty < E_e$, the photon will appear red-shifted far from the black hole, but how much will this redshift be? Clearly, right at the event horizon, $r = 2GM$ and $E_\infty = 0$ so light emitted there will not be visible, but how about just outside the event horizon? This is a pertinent question, since many black holes are believed to be surrounded by incandescent gas, so it may be possible to detect black holes by seeing the light emitted by this gas.

We'll consider the quantity $\varepsilon \equiv E_\infty/E_e$, so that

$$\varepsilon = 1-\sqrt{\frac{2GM}{r}} \tag{0.13}$$

The rate of change of ε is then

$$\frac{d\varepsilon}{dt} = \frac{GM}{r^2\sqrt{\frac{2GM}{r}}}\frac{dr}{dt} = \sqrt{\frac{GM}{2r^3}}\frac{dr}{dt} \tag{0.14}$$

For dr/dt, we use the four-velocity components for an object falling radially:

$$u_t' = \frac{dt}{d\tau} = e\left(1-\frac{2GM}{r}\right)^{-1} \tag{0.15}$$

$$u_z' = \frac{dr}{d\tau} = -\sqrt{e^2-\left(1-\frac{2GM}{r}\right)} \tag{0.16}$$

$$u_x' = \frac{d\phi}{d\tau} = 0 \tag{0.17}$$

$$u_y' = \frac{d\theta}{d\tau} = 0 \tag{0.18}$$

Since the laser is falling from rest at infinity $e = 1$ and we get

$$\frac{dr}{dt} = \frac{dr/d\tau}{dt/d\tau} \tag{0.19}$$

$$= -\left(1-\frac{2GM}{r}\right)\sqrt{\frac{2GM}{r}} \tag{0.20}$$

Therefore,

$$\frac{d\varepsilon}{dt} = -\sqrt{\frac{GM}{2r^3}}\left(1-\frac{2GM}{r}\right)\sqrt{\frac{2GM}{r}} \tag{0.21}$$

$$= -\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right) \tag{0.22}$$

$$= -\frac{GM}{r^2}\left(1-\sqrt{\frac{2GM}{r}}\right)\left(1+\sqrt{\frac{2GM}{r}}\right) \tag{0.23}$$

$$= -\frac{GM}{r^2}\left(1+\sqrt{\frac{2GM}{r}}\right)\varepsilon \tag{0.24}$$

Now if $r \approx 2GM$ (actually, it's just above $2GM$), then we can approximate this by

$$\frac{d\varepsilon}{dt} \approx -\frac{2GM}{(2GM)^2}\varepsilon = -\frac{\varepsilon}{2GM} \tag{0.25}$$

The solution of this differential equation is an exponential:

$$\varepsilon(t) = Ae^{-t/2GM} \tag{0.26}$$

for some constant A. From the equation above for dr/dt, we see that

$dr/dt \to 0$ as $r \to 2GM$, that is, t increases a large amount while r changes very little. Thus ε falls off very rapidly as we approach the event horizon, and any photons emitted just outside the event horizon will be red-shifted so much they will be undetectable. Further away from the black hole, of course, photons will escape and end up with a detectable energy at large distances; in fact the detection of X-rays emitted by material falling into the black hole is one way of detecting them.