PAINLEVÉ-GULLSTRAND COORDINATES: DERIVATION USING A LOCAL FLAT FRAME

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 15; Problem 15.1.

Earlier, we worked out the basis vectors in a locally flat frame for a freely falling observer near a black hole. These basis vectors are worked out by considering the four-velocity in two frames: the local, flat frame, and the Schwarzschild (S) frame. In particular, in the flat frame, $\mathbf{u}=\mathbf{o}_t=[1,0,0,0]$ so in the other frame, the four-velocity is the transformed time basis vector: $\mathbf{u}'=\mathbf{o}'_t$. Using this argument, we worked out $\mathbf{o}'_t$ in the S frame for a freely falling observer and got

$$\mathbf{o}'_t=\left[\left(1-\frac{2GM}{r}\right)^{-1},-\sqrt{\frac{2GM}{r}},0,0\right] \tag{1}$$

In the flat frame, we can write the interval between two events as $d\mathbf{s}=[d\tau,dx,dy,dz]$. In the Painlevé-Gullstrand system, the time coordinate $\mathring{t}$ is just the proper time of a freely falling observer, so $d\mathring{t}=d\tau$. Still in the flat frame, we have therefore

$$d\mathring{t}=-\eta_{ij}o_t^i ds^j=-\mathbf{o}_t\cdot d\mathbf{s} \tag{2}$$

since only the component o_t^t is non-zero, and $\eta_{tt}=-1$ in flat space. Since this is a scalar product, it has the same value in any coordinate system, such as the S system where we have

$$\begin{aligned} d\mathring{t} &= \mathbf{o}'_t\cdot d\mathbf{s}' && (3)\\ &= g_{ij}o_t^{\prime i}ds^{\prime j} && (4)\end{aligned}$$

In the S system, we have

$$d\mathbf{s}'=[dt,dr,d\theta,d\phi] \tag{5}$$

so

$$d\mathring{t} = -\left[-\left(1-\frac{2GM}{r}\right)\right]\left(1-\frac{2GM}{r}\right)^{-1}dt - \left(1-\frac{2GM}{r}\right)^{-1}\left(-\sqrt{\frac{2GM}{r}}\right)dr \tag{6}$$

$$= dt + \left(1-\frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}}dr \tag{7}$$

This agrees with the earlier result for Painlevé-Gullstrand.