

PAINLEVÉ-GULLSTRAND COORDINATES: DERIVATION USING A LOCAL FLAT FRAME

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 15; Problem 15.1.

Earlier, we worked out the basis vectors in a locally flat frame for a freely falling observer near a black hole. These basis vectors are worked out by considering the four-velocity in two frames: the local, flat frame, and the Schwarzschild (S) frame. In particular, in the flat frame, $\mathbf{u} = \mathbf{o}_t = [1, 0, 0, 0]$ so in the other frame, the four-velocity is the transformed time basis vector: $\mathbf{u}' = \mathbf{o}'_t$. Using this argument, we worked out \mathbf{o}'_t in the S frame for a freely falling observer and got

$$(0.1) \quad \mathbf{o}'_t = \left[\left(1 - \frac{2GM}{r} \right)^{-1/2}, -\sqrt{\frac{2GM}{r}}, 0, 0 \right]$$

In the flat frame, we can write the interval between two events as $d\mathbf{s} = [d\tau, dx, dy, dz]$. In the Painlevé-Gullstrand system, the time coordinate t° is just the proper time of a freely falling observer, so $dt^\circ = d\tau$. Still in the flat frame, we have therefore

$$(0.2) \quad dt^\circ = -\eta_{ij} o_t^i ds^j = -\mathbf{o}_t \cdot d\mathbf{s}$$

since only the component o_t^t is non-zero, and $\eta_{tt} = -1$ in flat space. Since this is a scalar product, it has the same value in any coordinate system, such as the S system where we have

$$(0.3) \quad dt^\circ = \mathbf{o}'_t \cdot d\mathbf{s}'$$

$$(0.4) \quad = g_{ij} o_t^i ds'^j$$

In the S system, we have

$$(0.5) \quad d\mathbf{s}' = [dt, dr, d\theta, d\phi]$$

so

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(0.6)

$$d\hat{t} = - \left[- \left(1 - \frac{2GM}{r} \right) \right] \left(1 - \frac{2GM}{r} \right)^{-1} dt - \left(1 - \frac{2GM}{r} \right)^{-1} \left(-\sqrt{\frac{2GM}{r}} \right) dr$$

(0.7)

$$= dt + \left(1 - \frac{2GM}{r} \right)^{-1} \sqrt{\frac{2GM}{r}} dr$$

This agrees with the earlier result for Painlevé-Gullstrand.