

PAINLEVÉ-GULLSTRAND METRIC: PHOTON PATHS INSIDE THE EVENT HORIZON

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 15; Box 15.3, Problem P15.3.

The Painlevé-Gullstrand (PG, also known as the global rain system) metric is a different way of describing empty space surrounding a spherically symmetric mass that avoids some of the problems encountered with the Schwarzschild metric (S metric). The PG metric is

$$(1) \quad ds^2 = - \left(1 - \frac{2GM}{r} \right) d\hat{t}^2 + 2\sqrt{\frac{2GM}{r}} d\hat{t} dr + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

where \hat{t} is the PG time coordinate, and is the proper time as measured by a clock falling along a radial line from rest at infinity. The \hat{t} coordinate, since it is measured as the proper time on a clock, is always a time coordinate, so the PG metric avoids the switch between distance and time coordinates that occurs in the S metric. It might seem that this causes problems for $r < 2GM$, since there the metric component $g_{\hat{t}\hat{t}} = - \left(1 - \frac{2GM}{r} \right)$ becomes positive, so it might seem that all the terms are positive, resulting in $ds^2 > 0$ for a time-like world line (contrary to what it should be). However, the off-diagonal metric component $g_{\hat{t}r} = 2\sqrt{\frac{2GM}{r}}$ comes to the rescue here, since for $r < 2GM$, if we have $dr < 0$ we can still get $ds^2 < 0$. That is, for a time-like world line, we must have $dr < 0$ once we cross the event horizon, which is the same condition that we got from the S metric, except there it was done by swapping the meanings of r and t .

For a radial world line $d\theta = d\phi = 0$ and we have

$$(2) \quad ds^2 = - \left(1 - \frac{2GM}{r} \right) d\hat{t}^2 + 2\sqrt{\frac{2GM}{r}} d\hat{t} dr + dr^2$$

which we can write as

$$(3) \quad \left(\frac{ds}{d\dot{t}}\right)^2 = -\left(1 - \frac{2GM}{r}\right) + 2\sqrt{\frac{2GM}{r}}\frac{dr}{d\dot{t}} + \left(\frac{dr}{d\dot{t}}\right)^2$$

Photons travelling radially will have $\left(\frac{ds}{d\dot{t}}\right)^2 = 0$, so we can find $\frac{dr}{d\dot{t}}$ by solving the quadratic equation on the RHS.

$$(4) \quad \frac{dr}{d\dot{t}} = \frac{1}{2} \left[-2\sqrt{\frac{2GM}{r}} \pm \sqrt{\frac{8GM}{r} + 4\left(1 - \frac{2GM}{r}\right)} \right]$$

$$(5) \quad = -\sqrt{\frac{2GM}{r}} \pm 1$$

The minus sign corresponds to an inward-moving photon, and the plus sign to an outward-moving one. Leaving aside the obvious problem that, for $r < 2GM$, both these speeds can exceed 1 in magnitude, we can solve this equation to find \dot{t} as a function of r . We get, for the inward-moving photon

$$(6) \quad -\int \frac{dr}{\sqrt{\frac{2GM}{r}} + 1} = \int d\dot{t}$$

Using Maple for the integral, we get

$$(7) \quad \dot{t} = -r + 2\sqrt{2GM}r - 4GM \ln \left[\sqrt{2GM} + \sqrt{r} \right] + C$$

where C is the constant of integration. If we start the photon at $r = 2GM$ at $\dot{t} = 0$, then

$$(8) \quad C = -2GM + 4GM \ln \left(2\sqrt{2GM} \right)$$

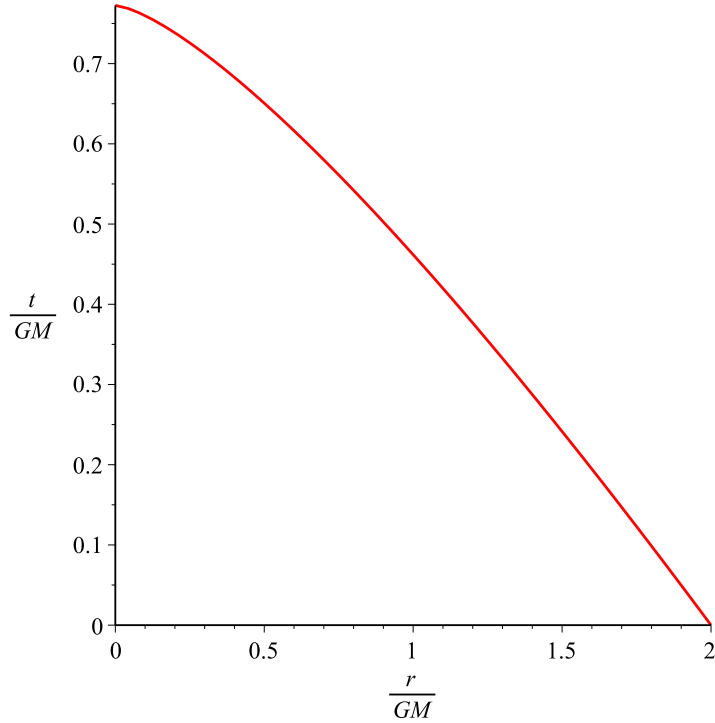
$$(9) \quad \frac{\dot{t}}{GM} = 2\sqrt{\frac{2r}{GM}} - \frac{r}{GM} - 2 - 4 \ln \left[\sqrt{2GM} + \sqrt{r} \right] + 4 \ln \left(2\sqrt{2GM} \right)$$

$$(10) \quad = 2\sqrt{\frac{2r}{GM}} - \frac{r}{GM} - 2 - 4 \ln \left[\frac{1}{2} \left(1 + \sqrt{\frac{r}{2GM}} \right) \right]$$

When $r = 0$, we get

$$(11) \quad \frac{\dot{t}}{GM} = -2 + 4 \ln 2 = 0.7726$$

A plot of $\frac{\dot{t}}{GM}$ versus $\frac{r}{GM}$ is as shown:



The particle starts off at $r/GM = 2$ and its world line travels up and to the left until it hits $r = 0$.

For an outward moving photon, we get

$$(12) \quad - \int \frac{dr}{\sqrt{\frac{2GM}{r} - 1}} = \int d\dot{t}$$

This time, we get for the integral

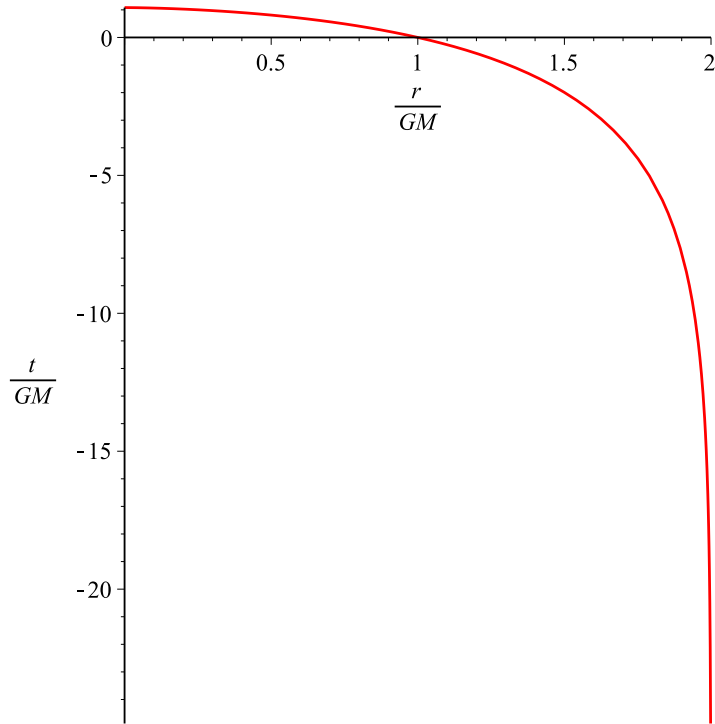
$$(13) \quad \dot{t} = r + 2\sqrt{2GM}r + 4GM \ln \left[\sqrt{2GM} - \sqrt{r} \right] + C$$

In this case, we can't start an outward-moving photon at $r = 2GM$, since this gives an infinite logarithm. In fact, if we emit an outward-moving photon at some $r < 2GM$, it will actually move *inward* (since we require $dr < 0$), although if we emit the photon near the event horizon (from the inside), it will take a very long time to move inwards from its starting point. To make things definite, suppose $\dot{t} = 0$ at $r = GM$. Then

$$(14) \quad C = -\left(1 + 2\sqrt{2}\right)GM - 4GM \ln\left(\left(\sqrt{2} - 1\right)\sqrt{GM}\right)$$

$$(15) \quad \frac{\dot{t}}{GM} = \frac{r}{GM} + 2\sqrt{\frac{2r}{GM}} - \left(1 + 2\sqrt{2}\right) + 4 \ln\left[\frac{\sqrt{2} - \sqrt{r/GM}}{\sqrt{2} - 1}\right]$$

A plot looks like this:



Again, the motion along the red curve is up and to the left. A photon emitted outward at $r = GM$ takes $\dot{t} = 1.0833GM$ to reach $r = 0$, while an inward moving photon takes $\dot{t} = 0.3108GM$ (which we get by evaluating \dot{t} for $r = GM$ in 7 and taking the difference between that and 0.7726).