PAINLEVÉ-GULLSTRAND METRIC: PHOTON PATHS INSIDE THE EVENT HORIZON

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 15; Box 15.3, Problem P15.3.

The Painlevé-Gullstrand (PG, also known as the global rain system) metric is a different way of describing empty space surrounding a spherically symmetric mass that avoids some of the problems encountered with the Schwarzschild metric (S metric). The PG metric is

$$ds^2 = -\left(1-\frac{2GM}{r}\right)d\mathring{t}^2 + 2\sqrt{\frac{2GM}{r}}d\mathring{t}\,dr + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{1}$$

where $\mathring{t}$ is the PG time coordinate, and is the proper time as measured by a clock falling along a radial line from rest at infinity. The $\mathring{t}$ coordinate, since it is measured as the proper time on a clock, is always a time coordinate, so the PG metric avoids the switch between distance and time coordinates that occurs in the S metric. It might seem that this causes problems for $r < 2GM$, since there the metric component $g_{\mathring{t}\mathring{t}} = -\left(1-\frac{2GM}{r}\right)$ becomes positive, so it might seem that all the terms are positive, resulting in $ds^2 > 0$ for a time-like world line (contrary to what it should be). However, the off-diagonal metric component $g_{\mathring{t}r} = 2\sqrt{\frac{2GM}{r}}$ comes to the rescue here, since for $r < 2GM$, if we have $dr < 0$ we can still get $ds^2 < 0$. That is, for a time-like world line, we must have $dr < 0$ once we cross the event horizon, which is the same condition that we got from the S metric, except there it was done by swapping the meanings of r and t.

For a radial world line $d\theta = d\phi = 0$ and we have

$$ds^2 = -\left(1-\frac{2GM}{r}\right)d\mathring{t}^2 + 2\sqrt{\frac{2GM}{r}}d\mathring{t}\,dr + dr^2 \tag{2}$$

which we can write as

$$\left(\frac{ds}{d\mathring{t}}\right)^2 = -\left(1-\frac{2GM}{r}\right) + 2\sqrt{\frac{2GM}{r}}\frac{dr}{d\mathring{t}} + \left(\frac{dr}{d\mathring{t}}\right)^2 \tag{3}$$

Photons travelling radially will have $\left(\frac{ds}{d\mathring{t}}\right)^2 = 0$, so we can find $\frac{dr}{d\mathring{t}}$ by solving the quadratic equation on the RHS.

$$\begin{aligned}
\frac{dr}{d\mathring{t}} &= \frac{1}{2}\left[-2\sqrt{\frac{2GM}{r}} \pm \sqrt{\frac{8GM}{r} + 4\left(1 - \frac{2GM}{r}\right)}\right] && (4)\\
&= -\sqrt{\frac{2GM}{r}} \pm 1 && (5)
\end{aligned}$$

The minus sign corresponds to an inward-moving photon, and the plus sign to an outward-moving one. Leaving aside the obvious problem that, for $r < 2GM$, both these speeds can exceed 1 in magnitude, we can solve this equation to find $\mathring{t}$ as a function of r. We get, for the inward-moving photon

$$-\int \frac{dr}{\sqrt{\frac{2GM}{r}} + 1} = \int d\mathring{t} \tag{6}$$

Using Maple for the integral, we get

$$\mathring{t} = -r + 2\sqrt{2GMr} - 4GM \ln\left[\sqrt{2GM} + \sqrt{r}\right] + C \tag{7}$$

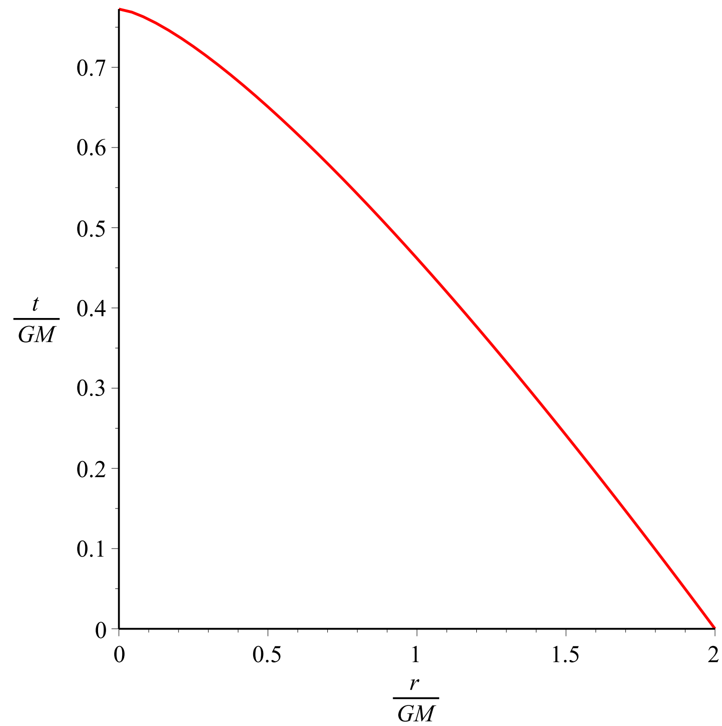
where C is the constant of integration. If we start the photon at $r = 2GM$ at $\mathring{t} = 0$, then

$$\begin{aligned}
C &= -2GM + 4GM \ln\left(2\sqrt{2GM}\right) && (8)\\
\frac{\mathring{t}}{GM} &= 2\sqrt{\frac{2r}{GM}} - \frac{r}{GM} - 2 - 4\ln\left[\sqrt{2GM} + \sqrt{r}\right] + 4\ln\left(2\sqrt{2GM}\right) && (9)\\
&= 2\sqrt{\frac{2r}{GM}} - \frac{r}{GM} - 2 - 4\ln\left[\frac{1}{2}\left(1 + \sqrt{\frac{r}{2GM}}\right)\right] && (10)
\end{aligned}$$

When $r = 0$, we get

$$\frac{\mathring{t}}{GM} = -2 + 4\ln 2 = 0.7726 \tag{11}$$

A plot of $\frac{\mathring{t}}{GM}$ versus $\frac{r}{GM}$ is as shown:



The particle starts off at $r/GM = 2$ and its world line travels up and to the left until it hits $r = 0$.

For an outward moving photon, we get

$$-\int \frac{dr}{\sqrt{\frac{2GM}{r}-1}} = \int d\mathring{t} \tag{12}$$

This time, we get for the integral

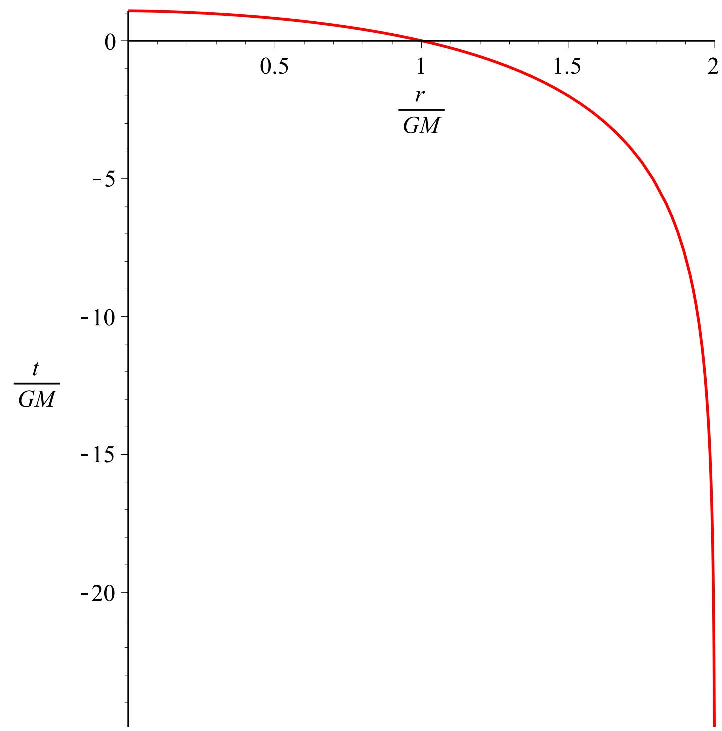
$$\mathring{t} = r + 2\sqrt{2GMr} + 4GM\ln\left[\sqrt{2GM}-\sqrt{r}\right] + C \tag{13}$$

In this case, we can't start an outward-moving photon at $r = 2GM$, since this gives an infinite logarithm. In fact, if we emit an outward-moving photon at some $r < 2GM$, it will actually move *inward* (since we require $dr < 0$), although if we emit the photon near the event horizon (from the inside), it will take a very long time to move inwards from its starting point. To make things definite, suppose $\mathring{t} = 0$ at $r = GM$. Then

$$C=-\left(1+2\sqrt{2}\right)GM-4GM\ln\left(\left(\sqrt{2}-1\right)\sqrt{GM}\right) \tag{14}$$

$$\frac{\mathring{t}}{GM}=\frac{r}{GM}+2\sqrt{\frac{2r}{GM}}-\left(1+2\sqrt{2}\right)+4\ln\left[\frac{\sqrt{2}-\sqrt{r/GM}}{\sqrt{2}-1}\right] \tag{15}$$

A plot looks like this:



Again, the motion along the red curve is up and to the left. A photon emitted outward at $r=GM$ takes $\mathring{t}=1.0833GM$ to reach $r=0$, while an inward moving photon takes $\mathring{t}=0.3108GM$ (which we get by evaluating $\mathring{t}$ for $r=GM$ in 7 and taking the difference between that and 0.7726) .